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Intersection and linking numbers of digital curves and application to the characterization of topology preservation in computer imagery

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Contents

Introduction

Topology is the field of mathematics which claims that a donuts is nothing but a special kind of coffee cup, provided this cup has an handle. Pattern recognition is the field of computer science which attempts to make a computer being able to discriminate a coffee cup from a donuts. Whereas these two motivations seam to be conflicting, it appears that digital topology is a fruitful field of investigations for pattern recognition. Indeed, since the early seventies, the definition of topological notions adapted to the digital framework have allowed to define, mathematically formalize and justify the algorithms used by image analysis and processing programs. Topology preserving operations do belong to this set of algorithms, and we do not need to recall the interest of (topology preserving) thinning algorithms for example in optical character recognition (OCR).

Now, more than the two dimensional case, an interesting field of investigation is the digital space $\mathbb{Z}^3$ to which many recent works are related. In this purpose, two main notions have been used in order to characterize topology preservation in $\mathbb{Z}^3$. The first one, the Euler characteristic, has been adapted from classical topology and graph theory. The second one, the fundamental group, has been adapted from algebraic topology to the digital framework ([45]). In this thesis, we are motivated by the study of the basic elements of the digital fundamental group : the *homotopy classes* of digital paths. Then, using some new tools dedicated to the study of such classes, we prove some new theorems which show that the digital fundamental group is a powerful tool for the study of topological properties in a digital space. Furthermore, we attempted in this work to provide some comprehensive proofs for all our results, proofs which only involves the basic notions classically used in digital topology, such as adjacency relations and of course integer arithmetic.

In a first part, we will introduce some notations and recall several mathematical notions of sets and graphs theory. These definitions, together with some basic mathematical notions like correspondences, maps and some elementary knowledge of groups theory, will allow the reader to understand all the results and their proofs which will be stated in the sequel. Then, we will introduce some classical definitions which are specific to digital topology and which have been either used or generalized in this work. Finally, we will end this part with the description, using practical examples, of the main problem which have motivated this thesis : the characterization of topology preservation in a digital space.

In a second part, we present a new framework for the study of the latter enunciated problem. This framework is constituted by the objects known in the literature as *digital boundaries* ([4, 40, 36, 92, 103]) and which will be simply called *digital surfaces* here, since no confusion is possible with some other kinds of digital surfaces which have been introduced by several authors ([76, 58, 67]). This kind of digital objects are practically interesting since they constitutes the basis of some image visualization and processing tools ([39, 40, 102, 52, 53]). Then, for both of these latter purposes, topology preservation problems raise when one deals with the definition of thinning algorithms in this field ([70]). Indeed, the use of thinning algorithms within surfaces today appears as an efficient tool for developing algorithms dedicated to the visualization and analysis of the underlying objects. We will recall in this part some previous results about the characterization of topology preservation within digital surfaces ([70]) and then introduce a new tool for proving theorems in this framework : the intersection number of paths. Some main properties of this new tool will be proved so that the intersection number can be used to distinguish the homotopy classes of paths. Then, using these properties, a new Jordan theorem for digital curves lying on a digital surface will be stated. Finally, this tool will allow the proof of a new theorem which states that the digital fundamental group is sufficient to characterize topology preservation within digital surfaces, except in a very particular and trivial case.

In a third part, we will no longer consider the field of digital surfaces but we will investigate the topology preservation problem in the digital space $\mathbb{Z}^3$. In this goal, we observe that the digital surfaces may be considered as an intermediate framework between the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$. In this last part, we provide the result of a work the intent of which was to achieve a rigorous formalization of a concise global characterization of *simple points* of $\mathbb{Z}^3$. These points are classically defined as points which can be removed from an object, preserving the topological properties of this object ([75, 11, 8, 54, 46]). More precisely, we will investigate in this part the "obvious" correspondence between the topological properties of a subset of $\mathbb{Z}^3$ and those of its background. Again, we are still interested by the characterization of these properties using the digital fundamental group. Then, defining a digital analogue to the linking number (basic invariant of knot theory) for closed paths in $\mathbb{Z}^3$ and proving its invariance property with some *continuous* deformation of the paths, we will be able to prove the correspondence previously mentioned. Indeed, the linking number, similarly to the intersection number introduced in the second part, allows to distinguish homotopy classes of closed paths in subsets of $\mathbb{Z}^3$. Finally, we will conclude with an open question which suggests a link between digital topology and knot theory.

Part I

Basic definitions and introduction to digital topology

Introduction to Part I

In this first part, we present the basic notations and definitions which will be used in the sequel of this document. After a very brief recall of several notions of sets and graphs theories, we will introduce few classical definitions of digital topology such as adjacency relations and connectivity. We will also introduce the definition of what could be called a *continuous deformation* of a digital path, properly : the homotopy relation for digital spaces. This latter relation will allow the definition of the digital fundamental group which was first introduced in digital topology by T.Y. Kong in [45]. Then, we will set and illustrate several definitions about topology preserving deformations of subsets of a digital space.

Chapter 1

Digital spaces

In this chapter we introduce some basic definitions which will be used in the sequel.

1.1 Sets, relations and paths

In this section, E is a set.

Notation 1.1 (complement, set of subsets) If $X \subset E$ we denote by $\overline{X}$ the complement of X in E, i.e the set of the elements of E which do no not belong to X. We denote by $P(E)$ the set of all the subsets of X.

Now, we recall the definitions of a binary relation and an equivalence relation on E.

Definition 1.1 (binary relation) *A* binary relation $\mathcal{R}$ on E *is a subset of* $E \times E$*.*

- $\mathcal{R}$ *is called* symmetric *if for all* $x \in E$ *and* $y \in E$*, then* $(x, y) \in \mathcal{R} \Rightarrow (y, x) \in \mathcal{R}$*.*
- $\mathcal{R}$ *is called* reflexive *if* $(e, e) \in \mathcal{R}$ *for any* $e \in E$*.*
- $\mathcal{R}$ *is called* transitive *if for all* a*,* b *and* c *in* E*, then* $(a, b) \in \mathcal{R}$ *and* $(b, c) \in \mathcal{R}$ *implies that* $(a, c) \in \mathcal{R}$*.*
- $\mathcal{R}$ *is called* anti-reflexive *if for all* $e \in E$*,* $(e, e) \notin \mathcal{R}$*.*
- $\mathcal{R}$ *is called* anti-symmetric *if for all* a *and* b *in* E*, then* $(a, b) \in \mathcal{R}$ *and* $(b, a) \in \mathcal{R}$ *implies* $a = b$*.*

We also recall the definition of the symmetric and transitive closure of a relation $\mathcal{R}$.

Definition 1.2 *If $\mathcal{R}$ is a binary relation on E, the* symmetric and transitive closure *of the relation $\mathcal{R}$ is the relation $\mathcal{R}_{st}$ defined by :*

$$Rel_{st} = \left\{ (x,y) \;\middle|\; \begin{array}{l} \exists (z_0, \ldots, z_l) \text{ such that for } i = 0, \ldots, l-1, \text{ either } (x_i, x_{i+1}) \in \mathcal{R} \text{ or} \\ (x_{i+1}, x_i) \in \mathcal{R}. \end{array} \right\}$$

Definition 1.3 *If $\mathcal{R}$ is a binary relation on E, the* transitive closure *of the relation $\mathcal{R}$ is the relation $\mathcal{R}_t$ defined by :*

$$Rel_t = \left\{ (x,y) \;\middle|\; \exists (z_0, \ldots, z_l) \text{ such that for } i = 0, \ldots, l-1, \text{ either } (x_i, x_{i+1}) \in \mathcal{R}. \right\}$$

Definition 1.4 (equivalence relation, equivalence class) *A binary relation $\mathcal{R}_{eq}$ on E is said to be an* equivalence relation *if it is reflexive, symmetric and transitive. If $\mathcal{R}_{eq}$ is an equivalence relation on E and $e \in E$, the* equivalence class of e following $\mathcal{R}_{\text{eq}}$, *denoted by $[e]_{\mathcal{R}_{eq}}$, is defined by :*

$$[e]_{\mathcal{R}_{eq}} = \{ \; x \in E \mid (x,e) \in \mathcal{R}_{eq} \; \}.$$

In the context of digital topology, the following notion of *adjacency relation* will be extensively used, as an analogue to the continuous notion of neighborhood.

Definition 1.5 (adjacency relation) *A binary relation $\mathcal{R}$ on E is said to be an* adjacency relation *if it is symmetric and anti-reflexive.*

In the sequel of this section, $\mathcal{R}$ is an adjacency relation on E.

Definition 1.6 ($\mathcal{R}-$adjacency, $\mathcal{R}-$neighborhood $N_{\mathcal{R}}(x)$) *Let $x \in E$ and $y \in E$. We say that x* and *y* are *$\mathcal{R}-$*adjacent *if $(x,y) \in \mathcal{R}$. Furthermore, if X is a subset of E we say that $x \notin X$ is $\mathcal{R}-$adjacent to X if there exists an element $y \in X$ such that $(x,y) \in \mathcal{R}$.*

*Given an element $x \in E$, we denote by $N_{\mathcal{R}}(x)$ and we call the $\mathcal{R}-$*neighborhood of *x the set of elements of E which are $\mathcal{R}-$adjacent to x.*

Remark 1.1 *The reader should be awarded about the possible confusion between the word "neighborhood" used here, and the same word used in topology. Indeed, although neighborhoods of an element as defined in topology always contains this element, a neighborhood in the sense of Definition 1.6 is the set of spels which are* close to a spel *according to a given adjacency relation, excluding the spel itself. Furthermore, the notation $N_{\mathcal{R}}(x)$ used here slightly differs from the standard one. Indeed, in many papers related to digital topology,* *the set $N_{\mathcal{R}}(x)$ contains x itself and a notation like $N^*_{\mathcal{R}}$ denotes the $\mathcal{R}-$neighborhood as defined here.*

Definition 1.7 (adjacency between sets) *Let X and Y be two subsets of E. We say that X and Y are $\mathcal{R}-$adjacent if there exists $x \in X$ and $y \in Y$ such that $(x, y) \in \mathcal{R}$.*

Definition 1.8 ($\mathcal{R}-$path, $\mathcal{R}-$connectivity, $\mathcal{R}-$connected components) *Let a and b be two elements of E. An $\mathcal{R}-$*path π from a to b with a length l in E *is a sequence $\pi = (x_0, \dots, x_l)$ of elements of E such that $a = x_0$, $b = x_l$ and for all $i \in \{0, \dots, l-1\}$ the elements x_i and x_{i+1} are $\mathcal{R}-$adjacent (i.e. $(x_i, x_{i+1}) \in \mathcal{R}$) or equal. The $\mathcal{R}-$path π is said to be* closed *if $x_0 = x_l$; and x_0 is then called the* base point *or* the extremity *of π. The path π is said to be* simple *if $x_i \neq x_j$ whenever $i \neq j$ (except when $\{i, j\} = \{0, l\}$ if π is closed). Two elements a and b of E are called $\mathcal{R}-$*connected in E *if there exists an $\mathcal{R}-$path from a to b in E. Obviously, the $\mathcal{R}-$connectivity is an equivalence relation on E and the equivalence classes of the $\mathcal{R}-$connectivity relation are called the $\mathcal{R}-$*connected components of E. *In other words, an $\mathcal{R}-$connected component of E is a maximal subset of E in which any two elements are $\mathcal{R}-$connected.*

We also define what we call a *local back and forth* in an $\mathcal{R}-$path.

Definition 1.9 (local back and forth) *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $\mathcal{R}-$path in E. We say that π has a* local back and forth *at the subscript k if $y_{k-1} = y_{k+1}$.*

Notation 1.2 In all this document, and in order to avoid heavy notations, for any *closed* path $\pi = (x_0, \dots, x_q)$ and any integer $i \in \{0, \dots, q\}$ the notation x_{i+1} [resp. x_{i-1}] should be read $x_{x+1 \bmod q}$ [resp. $x_{x-1 \bmod q}$], where we denote by $a + b \bmod q$ the only positive integer r such that $a + b = nq + r$ for some $n \in \mathbb{Z}$ and $r \in \{0, \dots, q-1\}$.

Definition 1.10 (change of parameter) *Let $\pi = (y_0, \dots, y_p)$ and $\pi' = (y'_0, \dots, y'_p)$ be two closed $\mathcal{R}-$paths with a length of p in E. The two paths π and π' are said to be* the same up to a change of parameter *if one of the two following properties hold :*

- *There exists $k_0 \in \{0, \dots, p\}$ such that for all $k \in \{0, \dots, p\}$ we have $y'_k = y_{k_0+k}$.*
- *There exists $k_0 \in \{0, \dots, p\}$ such that for all $k \in \{0, \dots, p\}$ we have $y'_k = y_{k_0-k}$.*

Now, we define the operation of concatenation between paths.

Definition 1.11 (concatenation of paths, inverse path) *Let $\pi = (y_0, \ldots, y_p)$ and $\pi' = (y'_0, \ldots, y'_q)$ be two $\mathcal{R}-$paths in E such that $y_p = y'_0$. We denote by $\pi.\pi'$ and we call* the concatenation of π and π' *the $\mathcal{R}-$path $(y_0, \ldots, y_{p-1}, y'_0, \ldots, y'_q)$. We also define and denote by π^{-1} the path $(y_p, y_{p-1}, y_{p-2}, \ldots, y_0)$ which is called the* inverse path of π.

Notation 1.3 If $\pi = (x_i)_{i=0,\ldots,p}$ is an $\mathcal{R}-$path in E, we denote by c^* the set :

$$\{y \in E \mid \exists i \in \{0, \ldots, p\},\ y = x_i\ \}.$$

Definition 1.12 (trivial path) *Let $a \in E$, the closed path $\pi = (a, a)$ from a to a (with a length 1) is called a* trivial path reduced to a.

Definition 1.13 (simple closed $\mathcal{R}-$curve) *A finite subset C of E is called a* simple closed $\mathcal{R}-$curve *if it is $\mathcal{R}-$connected and any point x of C is $\mathcal{R}-$adjacent to exactly two other elements of C. In this case, one can find a simple closed $\mathcal{R}-$path c in E such that $c^* = C$ which is called a* parameterization *of the curve C. Note that all such paths are all the same up to a change of parameter (see Definition 1.10). We call c a* parameterized simple closed $\mathcal{R}-$curve.

1.2 Digital images

In this section, we introduce the notions of *digital space* and *digital object*. First, we call a *digital space* any couple $(\mathcal{E}, \mathcal{R})$ where $\mathcal{R}$ is an adjacency relation. Note that *digital spaces* thus defined should not be confused with the digital spaces of classical topology. For the sake of simplicity, given $(\mathcal{E}, \mathcal{R})$, we also call $\mathcal{E}$ a digital space. An element x of a digital space $\mathcal{E}$ is called a *spel* (short for *sp*ace *el*ement).

In the sequel, unless otherwise stated, $(\mathcal{E}, \mathcal{R})$ is a digital space.

Definition 1.14 (digital image, object, background) *A* digital image $\mathcal{I}$ on $\mathcal{E}$ *is a couple $(\mathcal{E}, X)$ where X is a subset of $\mathcal{E}$. The set X is called a* digital object*. Elements of X are called* $1-$spels *or* black spels *whereas elements of $\overline{X}$ are called* $0-$spels *or* white spels*. The set $\overline{X}$ is called either the* background of the set X *or the* background of the digital image $\mathcal{I}$.

It becomes necessary to give some concrete examples of both digital spaces and digital images. In Figure 1.1 we have depicted using black points a subset of the set $\mathbb{Z}^2$. An element in $\mathbb{Z}^2$ is usually called a *pixel* as a short for *pic*ture *el*ement, and may be depicted using either circles or unit squares centered on points with integer coordinates of the Euclidean plane. In Figure 1.2, we have depicted, using unit cubes, a subset of $\mathbb{Z}^3$. Again, an element in $\mathbb{Z}^3$ is usually called a *voxel* as a short for *vo*lume *el*ement and is commonly depicted either by a point with integer coordinates, or by a unit cube centered on a point with integer coordinates.

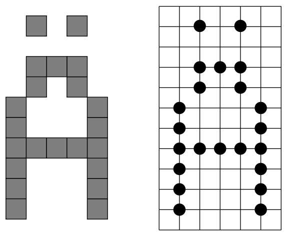


Figure 1.1: Two representations of a subset of $\mathbb{Z}^2$.

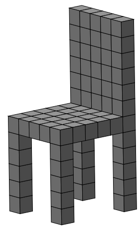


Figure 1.2: A view of an object in $\mathbb{Z}^3$.

1.3 Adjacency relations in $\mathbb{Z}^2$ and $\mathbb{Z}^3$

In this section, we define some adjacency relations in $\mathbb{Z}^2$ and $\mathbb{Z}^3$ which define digital spaces structures. Note that all the topological properties of a digital object one could study are immediately dependent on the choice of an adjacency relation (in fact two adjacency relations as explained in Section 1.4). Even if it is possible to define other kinds of adjacency relations for these two spaces, those presented here are the most commonly used.

First, let us introduce the following notation for digital spaces of the form $\mathbb{Z}^d$.

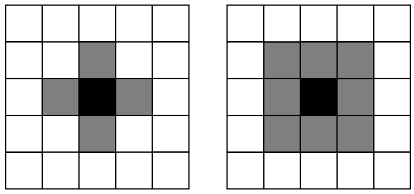
Notation 1.4 If $x \in \mathbb{Z}^d$, with $d \in \mathbb{N}$, we denote by x^i for $i \in \{1, \ldots, d\}$ the i^{th} coordinate of x, i.e. $x = (x^1, x^2, \ldots, x^d)$.

Definition 1.15 ($\mathcal{R}_4$ and $\mathcal{R}_8$) *We define the two adjacency relations $\mathcal{R}_4$ and $\mathcal{R}_8$ on $\mathbb{Z}^2$ as follows:*

$$\mathcal{R}_4 \quad = \quad \{\ (x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \ /\ |x^1 - y^1| + |x^2 - y^2| = 1\ \}$$

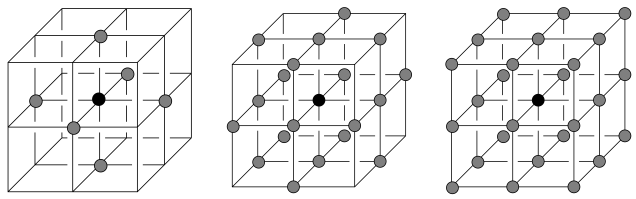
and,

$$\mathcal{R}_8 \quad = \quad \{\ (x, y) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \ /\ |x^1 - y^1|\,|x^2 - y^2| \leq 1\ \}$$



$n = 4$ $\qquad$ $n = 8$

Figure 1.3: A pixel x (in black) and the sets $N_n(x)$ for $n \in \{4, 8\}$ (in grey).



$n = 6$ $\qquad$ $n = 18$ $\qquad$ $n = 26$

Figure 1.4: A voxel x (black point) and the sets $N_n(x)$ for $n \in \{6, 18, 26\}$ (in grey).

Definition 1.16 ($\mathcal{R}_n$ for $n \in \{6, 18, 26\}$) *We define the three binary relations $\mathcal{R}_6$, $\mathcal{R}_{18}$ and $\mathcal{R}_{26}$ on $\mathbb{Z}^3$ as follows:*

$$\begin{array}{rcl}
\mathcal{R}_{26} & = & \{\ (x, y) \in \mathbb{Z}^3 \times \mathbb{Z}^3\ /\ Max[|x^1 - y^1|, |x^2 - y^2|, |x^3 - y^3|] = 1\ \} \\
\mathcal{R}_{18} & = & \{\ (x, y) \in \mathbb{Z}^3 \times \mathbb{Z}^3\ /\ |x^1 - y^1| + |x^2 - y^2| + |x^3 - y^3| \leq 2\ \}\ \cap\ \mathcal{R}_{26} \\
\mathcal{R}_6 & = & \{\ (x, y) \in \mathbb{Z}^3 \times \mathbb{Z}^3\ /\ |x^1 - y^1| + |x^2 - y^2| + |x^3 - y^3| = 1\ \}
\end{array}$$

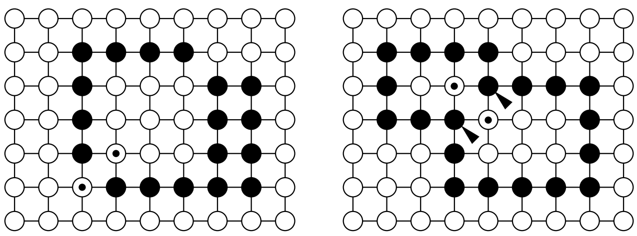
Notation 1.5 ("$n-$" notation) In order to improve readability, and since the latter adjacency relations will be used very often, we will abbreviate $\mathcal{R}_n$ to simply n in all the notations which have been introduced in the previous section. For example, $6-$adjacency is a short for $\mathcal{R}_6-$adjacency and $N_{26}(x)$ is a short for $N_{\mathcal{R}_{26}}(x)$.

The different neighborhoods (see Definition 1.6) associated with the adjacency relations of Definitions 1.15 and 1.16 are depicted respectively in Figures 1.3 and 1.4.

1.4 Complementarity between adjacencies

Now, we will justify the need of the use of two complementary adjacency relation when dealing with topological properties of digital images.

Indeed, the aim of digital topology is the study of topological properties of digital images. In this context, connectivity is obviously a fundamental notion. However, in order to prevent some topological paradoxes, we must add some restrictions on the use of adjacency relations when dealing with objects together with their complement. These paradoxes appear for example when we try to define *holes* in objects of $\mathbb{Z}^2$ and *cavities* in objects of $\mathbb{Z}^3$. The holes in $\mathbb{Z}^2$ and cavities in $\mathbb{Z}^3$ of an object are expected to be the finite connected components of the complement of this object. However, if the same adjacency relation is used to define connected component of an object X and connected components of its complement $\overline{X}$, then the previous definition for holes leads to some situations which are not acceptable. In Figure 1.5(a), the set $X \subset \mathbb{Z}^2$ of black pixels is $8-$connected and is expected to surround a hole made of white pixels. But since the two dotted pixels are $8-$adjacent, it is then possible to define a hole only as a $4-$connected component of white pixels. By the same way, we have depicted in Figure 1.5(b) a simple closed $4-$path. Since the two black pixels marked by an arrow are not $4-$adjacent, the path is expected to surround a single connected component of its complement. This is true if the two dotted pixels are themselves adjacent, that is to say if we consider the complement of the $4-$path with the $8-$connectivity. According to Kong ([49]), these consideration were first touched on in [23].



(a) (b)

Figure 1.5: Two objects of $\mathbb{Z}^2$.

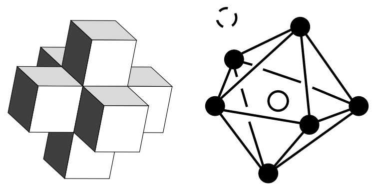


Figure 1.6: A topological sphere in 18 or $26-$connectivity.

Now, same considerations can be applied for subsets of $\mathbb{Z}^3$ as depicted in Figure 1.6. Let X be the object made of all the visible voxels in left part of the figure. Then, if the black voxels are considered as 26 or $18-$connected, the enclosed white voxel is expected to be *isolated* (i.e. not adjacent to any other white voxel) ; in such a way that the set of black voxels constitutes a topological sphere with one cavity. Obviously, this is true only if we use the $6-$adjacency relation for elements of $\overline{X}$.

Finally, we sum up these considerations by the use of couples $(\mathcal{R}, \overline{\mathcal{R}})$ of adjacency relations, the $\mathcal{R}-$adjacency being used for an object and the $\overline{\mathcal{R}}-$adjacency relation being used for its complement. For digital images on $\mathbb{Z}^2$, we will use $(n, \overline{n}) \in \{(4,8), (8,4)\}$ and for digital images on $\mathbb{Z}^3$, we will use $(n, \overline{n}) \in \{(6,26), (6+,18), (26,6), (18,6+)\}$. Note that we denote by 6+ the $6-$adjacency relation associated to the $18-$adjacency relation (note that $\mathcal{R}_{6+} = \mathcal{R}_6$ following Definition 1.16). Indeed, this will allow us to shorten notations since the data of the adjacency relation used for the object then implies the one used for its complement (if $n = 26$ then $\overline{n} = 6$, if $n = 6+$ then $\overline{n} = 18$). In the sequel, the latter enumerated couples $(n, \overline{n})$ are said to be couples of *complementary adjacency relations*.

More generally, we suppose in the sequel that the data of an adjacency relation $\mathcal{R}$ associated to an object X implies the use of a unique adjacency relation $\overline{\mathcal{R}}$ for its complement. Thus, the notation $\overline{\mathcal{R}}$ makes sense. For this purpose, each time we introduce some notation for an adjacency relation (e.g. (6+)), we should precise once for all the associated complementary adjacency relation.

Now, given a couple $(\mathcal{R}, \overline{\mathcal{R}})$ of complementary adjacency relations, we will also use the following definitions.

Definition 1.17 (black/white $\mathcal{R}-$components) *Let $\mathcal{I} = (\mathcal{E}, X)$ be a digital image and $\mathcal{R}$ be an adjacency relation on $\mathcal{E}$. An $\mathcal{R}-$connected component of X [resp. an $\overline{\mathcal{R}}-$connected component of $\overline{X}$] will be called a* black component *[resp.* white component *or* background component*].*

The following definition should not be confused with the notion of tunnels which will be introduced in the sequel.

Definition 1.18 ($\mathcal{R}-$cavity) *Let $\mathcal{I} = (\mathcal{E}, X)$ be a digital image and $\mathcal{R}$ be the adjacency relation on $\mathcal{E}$. A finite $\overline{\mathcal{R}}-$connected component of $\overline{X}$ will be called an $\mathcal{R}-$*cavity *of X. A cavity in $\mathbb{Z}^2$ is also called a* hole*.*

Definition 1.19 ($\mathcal{R}-$isolated spel, $\mathcal{R}-$interior spel) *Let X be a subset of $\mathcal{E}$ and $x \in X$. We say that the spel x is an $\mathcal{R}-$*isolated spel of *X when $N_{\mathcal{R}}(x) \cap X = \emptyset$; and x is called an $\mathcal{R}-$*interior spel of *X if $N_{\overline{\mathcal{R}}}(x) \cap \overline{X} = \emptyset$.*

Chapter 2

Basic notions for digital topology

In this chapter we introduce a few notions which have been considered by several authors in the framework of digital topology. For some of them, we have attempted to show how these notions are close or derived from analogous ones in the field of classical topology.

All these notions are presented here for two main reasons. First, they have proved to be useful for studying topological properties of digital images. On the other hand, tools which are developed in Parts II and III make use of all these notions and sometimes generalize them.

2.1 Index, winding number and Jordan Theorem

We recall here the definitions of two tools which have already been used in order to prove some fundamental results in the field of digital topology.

- The first one, which we will call the *index*, has been introduced by A. Rosenfeld in [87] and used to prove a digital Jordan theorem (Theorem 4 in further subsection 2.1.3) for digital pictures (i.e. digital images on $\mathbb{Z}^2$). Indeed, this number characterizes the fact that a given pixel belongs either to the *inside* or to the *outside* of a simple closed $n-$curve.

- The second one, the *2d winding number* has been introduced by R. Malgouyres ([58]) to provide a way to distinguish connected components of the complement of any closed path in $\mathbb{Z}^2$, and especially to characterize the *outside* of such a path.

2.1.1 Index of a pixel

This number is associated to a pixel x in the complement of a simple closed $n-$path $c = (x_0, \ldots, x_q)$ in $\mathbb{Z}^2$ for $n \in \{4, 8\}$.

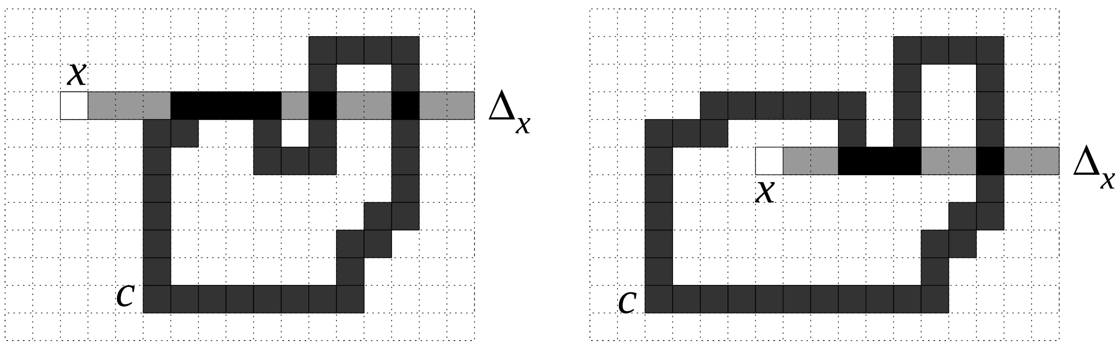
Let $x = (a, b) \in \mathbb{Z}^2$ be a pixel in the complement $\overline{c^*}$ of the pixels of c and $\Delta_x = \{(a + k, b) \mid k \in \mathbb{N}\}$. The right half line Δ_x thus defined (see Figure 2.1) intersects c on several *intersection intervals*. We define $\Lambda_{x,c}$ as the set of intersection intervals between c and Δ_x :

$$\Lambda_{x,c} = \{(k_1, k_2) \mid \{x_{k_1-1}, x_{k_2+1}\} \cap \Delta_x = \emptyset \text{ and } x_k \in \Delta_x \text{ for } k_1 \leq k \leq k_2\}.$$

Now, for any intersection interval $\lambda = (k_1, k_2) \in \Lambda_{x,c}$, we say that Δ_x *touches* c at λ if $(x^2_{k_1-1} - b)(x^2_{k_2+1} - b) = 1$ (i.e. x_{k_1-1} and x_{k_2+1} are on the same "side" of Δ_x) and we say that Δ_x *crosses* c at λ if $(x^2_{k_1-1} - b)(x^2_{k_2+1} - b) = -1$.

In [87], Rosenfeld defined the *inside* and *outside* sets associated with the simple closed n-curve c as follows :

- If $\Delta_{x,c}$ crosses c an odd number of times then x belongs to the *inside* of c,
- otherwise x belongs to the *outside* of c.



(a) The half line Δ_x *crosses* c twice, i.e. x is *outside* c.

(b) The half line Δ_x *crosses* c only one time, i.e. x is *inside* c.

Figure 2.1: Definition of inside and outside pixels in $\overline{c^*}$ when c is a simple closed $4-$curve.

Then, Rosenfeld first proved the following Proposition.

Proposition 2.1 ([87]) *The inside and the outside sets of a simple closed* $4-$*curve with a length greater than* 4 *are both nonempty.*

And then, he proved that :

Proposition 2.2 ([87]) *Let c be a simple closed $4-$curve with a length greater than 4. If x is a pixel in the inside of c and y is a pixel in the outside of c, then, any $8-$path from x to y intersects c.*

These two latter propositions state nothing but the fact that the complement of the simple closed curve c cannot be $8-$connected since there must exist (Proposition 2.1) two pixels x and y which are not $8-$connected in $\overline{c^*}$ (Proposition 2.2). Later, in [88], Rosenfeld achieved the proof of the so called Jordan theorem which states that the complement of a simple closed $4-$curve with a length greater than 4 is made of two $8-$connected components, which obviously coincide with its inside and outside sets (see subsection 2.1.3).

With less concise but not more difficult proofs, all the previous propositions given in this subsection are obviously valid in the case when c is a simple closed $8-$curve, with the convenient definitions.

Now, this simple but useful tool can be improved in order to distinguish connected components of the complement of any closed $n-$path (for $n \in \{4, 8\}$), not only simple closed paths. This leads to the definition of the winding number of a curve around a pixel in $\mathbb{Z}^2$ which has been used by R. Malgouyres.

2.1.2 The winding number

Motivation : a Jordan theorem in 3D

In [58, 61], R. Malgouyres has used the *winding number*, the definition of which is recalled in this section. He needed this tool to prove a Jordan theorem, as Rosenfeld did for simple closed curves in $\mathbb{Z}^2$, but this time for a kind of digital closed surfaces of $\mathbb{Z}^3$ which have been defined in [58, 61] : the $\mathcal{MA}-$surfaces. Such a surface is a set of voxels the $26-$neighborhood of which satisfies a so called *local condition* (readers interested by the different characterizations of surfaces made of voxels may refer for example to [76], [61], [12], [28], [68] or [68]). Then, it was proved that any $26-$connected set of voxels such that each one satisfies the latter local condition does separates its background in two $6-$connected components.

Like for the 2D case, a first step was to define *interior* and *exterior* voxels in the complement of a closed surface S. Then, an analogue to the method described in previous subsection is to count the number of times an half line D_y from a pixel y and parallel to one of the coordinates axes, crosses the surface. In the 3D case, we must still distinguish a *crossing* intersection between D_y and the surface S from a *touching* one. These two kinds of intersections are depicted in upper images of Figure 2.2(a) and 2.2(b). In order to distinguish these two kinds of intersections, we first consider the set A of voxels of S which are 26−adjacent to this intersection (see middle images in Figure 2.2). Then, Malgouyres proved that it is possible to give a periodic but unique parameterization, up to an orientation, of the set A ([61] and [59]) and, by the same way, of its projection on a 2D digital plane orthogonal to the line D_y. Because of the definition of the surface considered, this projection can be parameterized as an 8−path. Given this closed 8−path, which is obviously included in a 3×3 grid, the fact that the line crosses or touches the surface is equivalent to the fact that the central pixel of the 3×3 neighborhood (bottom images of Figures 2.2(a) and 2.2(b)) is either *inside* or *outside* the projected closed 8−path.

Now, a full definition of the inside and outside sets of a non simple closed 8−path must be given.

Winding number

The definition of the winding number is quite similar with the index one defined in subsection 2.1.1 but this time counts not only the number of times an half line crosses the path but the number of *oriented transversal intersections*.

Let $x = (a, b)$ be a pixel in the complement of a (not necessarily simple) closed 8−path $c = (x_0, \ldots, x_q)$ in $\mathbb{Z}^2$ for $n \in \{4, 8\}$. Let Δ_x^α be a half line (a *ray*) for $\alpha \in \{1, 2, 3, 4\}$ defined as follows :

$$\Delta_x^1 = \{(a + k, b) \mid k \in \mathbb{N}\}.$$

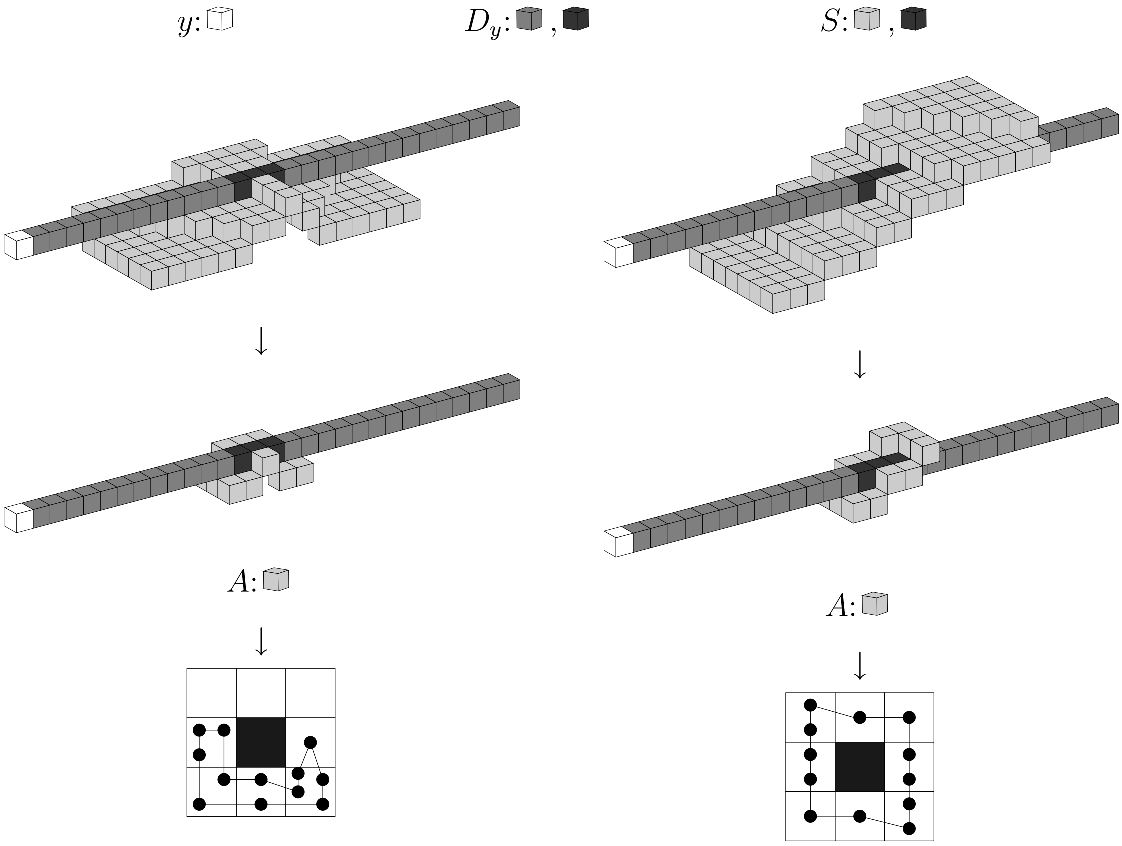
$$\Delta_x^2 = \{(a, b + k) \mid k \in \mathbb{N}\}.$$

$$\Delta_x^3 = \{(a - k, b) \mid k \in \mathbb{N}\}.$$

$$\Delta_x^4 = \{(a, b - k) \mid k \in \mathbb{N}\}.$$

The half line Δ_x^α thus defined (see Δ_x^1 in Figure 2.4) intersects c on several *intersection intervals*. We define $\Lambda_{x,c}^\alpha$ as the set of intersection intervals between c and Δ_x^α :

$$\Lambda_{x,c}^\alpha = \{(k_1, k_2) \mid \{x_{k_1-1}, x_{k_2+1}\} \cap \Delta_x^\alpha = \emptyset \text{ and } \forall k_1 \leq k \leq k_2,\, x_k \in \Delta_x^\alpha\ \}.$$



(a) The half line D_y touches the surface.

(b) The half line D_y crosses the surface.

Figure 2.2: Upper images : Two kinds of intersections between an half line D_y from y and the part of a $\mathcal{MA}-$surface. Middle images : The set A of voxels which are $26-$adjacent to some voxels of the intersection. Bottom images : The 2d projection of the set A on a plane orthogonal to D_y can be parameterized as a (not necessarily simple) closed $8-$path.

Now, for any intersection interval $\lambda = (k_1, k_2) \in \Lambda_{x,c}$, we say that λ is *tangent* intersection between c and Δ_x^α if $(x^2_{k_1-1} - b) \times (x^2_{k_2+1} - b) = 1$ (i.e. x_{k_1-1} and x_{k_2+1} are on the same "side" of Δ_x) and we say that λ is a *transversal intersection* between c and Δ_x if $(x^2_{k_1-1} - b) \times (x^2_{k_2+1} - b) = -1$. In this latter case, we associate to λ either the value $+1$ or -1 depending on the orientation of the intersection following the sign conventions given by Figure 2.3. Then, the winding number $W^\alpha_{x,c}$ is nothing but the sum of the *contributions* of the intersections between Δ_x^α, being 0, +1 or -1 respectively for tangent, positive transversal and negative transversal intersections (see Figure 2.5 for some examples). Then, the following theorem has been proved in [58].

Theorem 1 ([58]) *Let c be a closed $8-$path in $\mathbb{Z}^2$ and $x \in \mathbb{Z}^2 \setminus c^*$. Then, $W^1_{x,c} = W^2_{x,c} = W^3_{x,c} = W^4_{x,c}$.*

This latter theorem allows to abbreviate $W^\alpha_{x,c}$ to $W_{x,c}$ since its value does not depend on the choice of an half line Δ_x^α for $\alpha \in \{1, 2, 3, 4\}$ and shows that the winding number is an intrinsic notion. Furthermore, the following theorem sates that the winding number may allow to distinguish pixels which do not belong to a same $4-$connected component of the complement of a closed $8-$path. However, it does not provide a necessary condition for this purpose.

Theorem 2 ([58]) *Let c be a closed $8-$path in $\mathbb{Z}^2$ and $\{x, y\} \subset \mathbb{Z}^2 \setminus c^*$. If x and y are $4-$connected in $\overline{c^*}$ then $W^\alpha_{x,c} = W^\alpha_{y,c}$.*

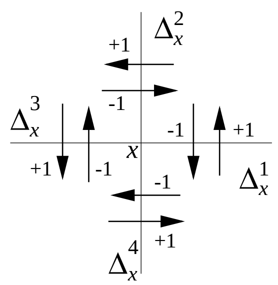
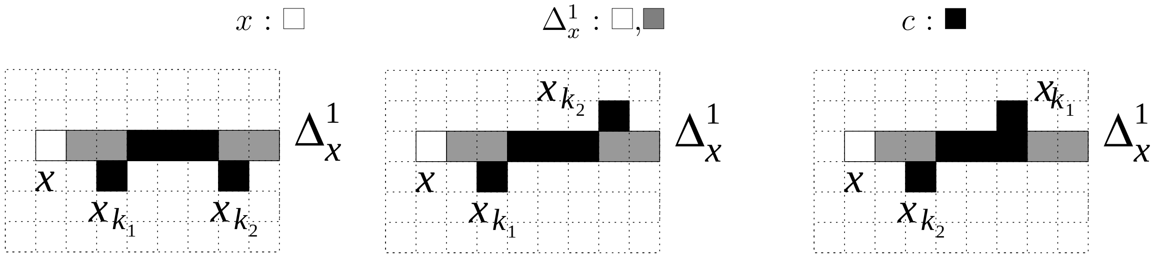


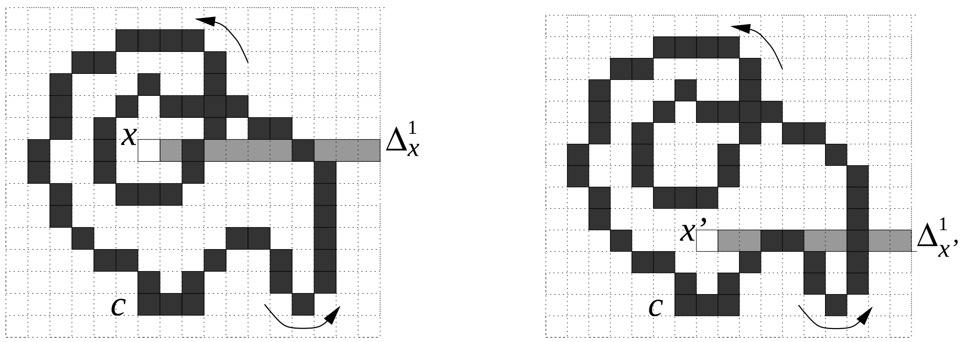
Figure 2.3: Sign conventions for transversal intersections.

In Chapter 6 we introduce the *intersection number* between digital paths which lie on a digital surface (this kind of a digital space will be defined in Chapter 4). Then, some new properties of the two dimensional winding number can be deduced from the properties of the intersection number (see Chapter 8). Theorem 2 will thus appear as an immediate corollary of some more general properties of the intersection number.



(a) Tangent intersection. (b) Positive intersection. (c) Negative intersection.

Figure 2.4: Tangent and transversal intersections.



(a) $W_{x,c} = +2$ (b) $W_{x,c} = +1$

Figure 2.5: Two examples of winding numbers.

2.1.3 A digital Jordan Theorem

In this subsection we recall a simple but important result due to Rosenfeld about a digital Jordan curve theorem. This very simple property is the discrete analogue of the well known similar theorem about simple closed curves in $\mathbb{R}^2$. This can be seen as a basic example of a classical result of topology which can be transposed in the digital framework. In Part II (Chapter 7), we will prove a new Jordan Theorem in the field of digital surfaces using a new tool, the *intersection number*, in such a way that the 2D Jordan Theorem becomes a restriction of this new theorem to the case of the digital plane $\mathbb{Z}^2$.

We recall that a simple closed curve in $\mathbb{R}^2$ is a continuous map f from the real closed interval $[0,1]$ to $\mathbb{R}^2$ such that $f(0) = f(1)$ and the restriction of f on $[0,1[$ is one-to-one.

Theorem 3 (Jordan curve theorem) *If f is a simple closed curve in $\mathbb{R}^2$, then $\mathbb{R}^2 \setminus f([0,1])$ has two connected components, one of which is bounded and the other is unbounded.*

In [87] and [88], A. Rosenfeld has proved the following theorem :

Theorem 4 ([88]) *If c is a digital simple closed $n-$curve in $\mathbb{Z}^2$ (with a length greater than 4 if $n = 4$), then $\mathbb{Z}^2 \setminus c^*$ has two $\overline{n}-$connected components, one of which is bounded and the other is unbounded.*

2.2 Homotopy

In this section, we deal with another very useful notion of algebraic topology which can be defined in the digital context : the homotopy of paths. First, we recall the definition of an *homotopy* in algebraic topology and then we will show how this notion can be defined with consistence in the digital framework. Provided this definition, it will be possible to introduce in the next section the *digital fundamental group* which is one of the most important subject of investigations of this thesis.

2.2.1 Homotopy in classical topology

We first state few definitions in the field of classical point set topology. However, since our purpose is not to state here all the notions of classical topology, these notions may be found in [43, 71], like for example the definition of a continuous map between two topological spaces which will be used in the sequel.

Definition 2.1 (topological space) *A topological space is a couple $(E, \mathcal{O})$ where E is a set and $\mathcal{O} \subset P(E)$ is such that :*

i) $\emptyset \in \mathcal{O}$ *and* $E \in \mathcal{O}$.

ii) The union of any collection of elements of $\mathcal{O}$ belongs to $\mathcal{O}$.

iii) The intersection of any two elements of $\mathcal{O}$ belongs to $\mathcal{O}$.

Elements of $\mathcal{O}$ are called open sets *of the topological space $(E, \mathcal{O})$. The set $\mathcal{O}$ is also called a* topology on E.

Definition 2.2 (path) *Let $(E, \mathcal{O})$ be a* topological space *and let x and y be two elements of E. A* path f from x to y in E *is a continuous map from $[0, 1]$ to E such that $f(0) = x$ and $f(1) = y$. The path f is said to be* closed *if $f(0) = f(1)$. A* trivial path *in E is a path f_0 such that f_0 is constant (i.e $\exists x \in E$, $\forall t \in [0, 1]$, $f_0(t) = x$).*

Definition 2.3 (catenation of paths) *Let E be a* topological space *and let f and g be two paths in E such that $f(1) = g(0)$. We define the catenation of the two paths f and g and we denote by $f.g$ the following path in E :*

$$\begin{array}{rrcl} f.g: & [0,1] & \longrightarrow & E \\ & s \in [0, \frac{1}{2}] & \longmapsto & f(2s) \\ & s \in (\frac{1}{2}, 1] & \longmapsto & g(2(s - \frac{1}{2})) \end{array}$$

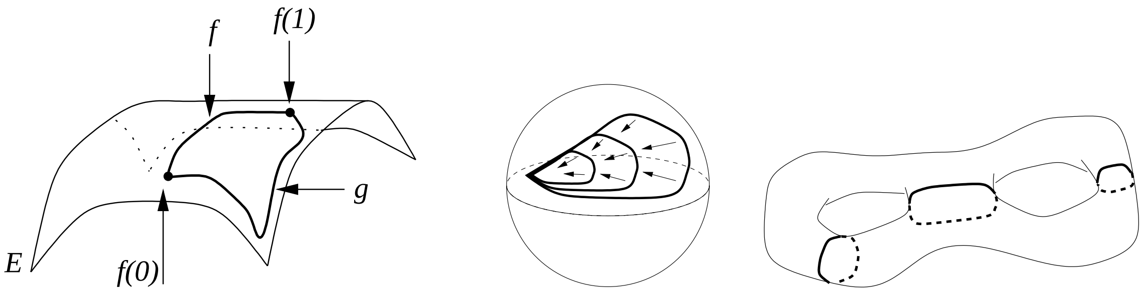
Definition 2.4 (homotopy with fixed extremities) *Let f and g be two paths in a topological space $(E, \mathcal{O})$ with same extremities (i.e. $f(0) = g(0)$ and $f(1) = g(1)$). We say that f and g are* homotopic with fixed extremities in E *if there exists a continuous map $H : [0, 1] \times [0, 1] \longrightarrow E$ such that for any $s \in [0, 1]$, we have $H(0, s) = f(s)$, $H(1, s) = g(s)$ and for any $t \in [0, 1]$, $H(t, 0) = f(0) = g(0)$ and $H(t, 1) = f(1) = g(1)$.*

Definition 2.5 (homotopy with base point) *Let $(E, \mathcal{O})$ be a topological space and $x \in E$. Two closed paths f and g from x to x in E are said* homotopic with base point x *if they are homotopic with fixed extremities in E.*

Remark 2.1 *If we denote by $A_B(E)$ the set of closed paths in E from the point B to B (which is called the* base point*), then, the relation of homotopy with base point B is an equivalence relation on $A_B(E)$.*

An important definition is the definition of a *simply connected space.*

Definition 2.6 (simply connected space) *A path-connected topological space $(E, \mathcal{O})$ is called* simply connected *if for any base point $x \in E$, every closed path from x to x in E is homotopic with base point x to a trivial path.*



(a) Two homotopic paths f and g in E.

(b) A simply connected space (sphere in $\mathbb{R}^3$).

(c) A not simply connected space (closed surface of $\mathbb{R}^3$).

Figure 2.6: Homotopy in a topological space and simply connected space.

The following lemma will be used in the next section in order to justify the definition of homotopy for digital paths. This lemma comes from the very definition of homotopy and simply connected sets and is illustrated in Figure 2.7.

Lemma 2.3 *Let f and g be two paths in a topological space E. Let V be a simply connected subset of E. Furthermore, let $I = [a, b] \subset [0, 1]$ such that $\forall x \in [0, a] \cup [b, 1]$, we have $f(x) = g(x)$; and $\forall x \in [a, b]$, we have $f(x) \in V$ and $g(x) \in V$. Then f and g are homotopic with fixed extremities in E.*

Sketch of proof : First, we must recall a classical result of algebraic topology which states that the catenation of paths is compatible with the relation of homotopy with fixed extremities. In other words, for any maps f, f', g and g' such that $f(1) = f'(1) = g(0) = g'(0)$, if f and f' are homotopic with fixed extremities in E then $f.g$ and $f'.g$ are homotopic with fixed extemities in E; and if g and g' are homotopic with fixed extemities in E then $f.g$ and $f.g'$ are homotopic with fixed extemities in E.

Now, we can define the following paths in E :

$$\begin{array}{rcrcl} f_1 & : & [0,1] & \longrightarrow & E \\ & & s & \longmapsto & f(sa) \\ f_2 & : & [0,1] & \longrightarrow & E \\ & & s & \longmapsto & f(a+s(b-a)) \end{array} \qquad \begin{array}{rcrcl} f_3 & : & [0,1] & \longrightarrow & E \\ & & s & \longmapsto & f(b+s(1-b)) \\ g_2 & : & [0,1] & \longrightarrow & E \\ & & s & \longmapsto & g(a+s(b-a)) \end{array}$$

It is readily seen that $f = f_1.f_2.f_3$ and $g = f_1.g_2.f_3$, furthermore f_2 and g_2 are continuous maps from $[0,1]$ to V which is simply connected. From Definition 2.6, it follows that f_2 and g_2 are homotopic with fixed extemities in V (and so in E). Finally, we obtain that f and g are homotopic with fixed extemities in E. $\square$

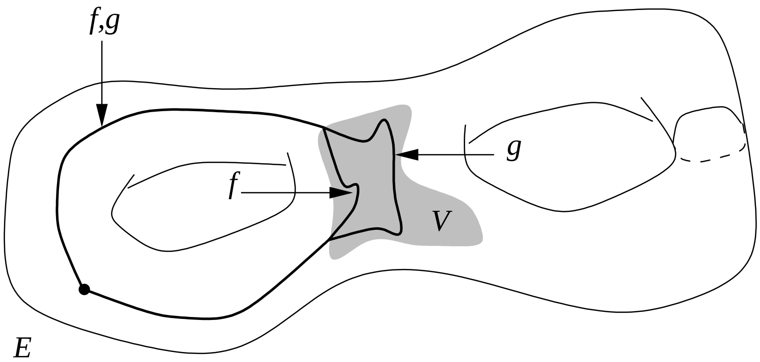


Figure 2.7: Two maps f and g which are the same but in a simply connected subset V of E.

2.2.2 Homotopy for digital paths

In this subsection, we show that the definition of homotopy between paths in a topological space can be transposed in the discrete context. First of all, a digital $\mathcal{R}-$path with a length l in an object X as defined in Section 1.1 can also be defined as a map from the integer interval $\{0, \ldots, l\}$ to X such that two consecutive integers are sent on $\mathcal{R}-$adjacent spels of X. Now, we have to define the homotopy relation between paths. Here, we will give an intuitive justification of the definition which will follow. For a detailed definition and justification, see [45].

First, and since examples will be given for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$, we need to define the two following small subsets of $\mathbb{Z}^3$:

Definition 2.7 (2×2 square) *A* 2×2 square *in* $\mathbb{Z}^3$ *is one of the following sets for* $x = (x^1, x^2, x^3) \in \mathbb{Z}^3$ *:*
$\{(x^1, x^2, x^3), (x^1 + 1, x^2, x^3), (x^1, x^2 + 1, x^3), (x^1 + 1, x^2 + 1, x^3)\}$,
$\{(x^1, x^2, x^3), (x^1 + 1, x^2, x^3), (x^1, x^2, x^3 + 1), (x^1 + 1, x^2, x^3 + 1)\}$ *and*
$\{(x^1, x^2, x^3), (x^1, x^2 + 1, x^3), (x^1, x^2, x^3 + 1), (x^1, x^2 + 1, x^3 + 1)\}$.

Definition 2.8 (2×2×2 cube) *A* 2×2×2 cube *in* $\mathbb{Z}^3$ *is a set of the following form for* $x = (x^1, x^2, x^3) \in \mathbb{Z}^3$ *:*
$\{(x^1, x^2, x^3), (x^1 + 1, x^2, x^3), (x^1 + 1, x^2, x^3), (x^1 + 1, x^2 + 1, x^3), (x^1, x^2, x^3 + 1),$
$(x^1 + 1, x^2, x^3 + 1), (x^1 + 1, x^2, x^3 + 1), (x^1 + 1, x^2 + 1, x^3 + 1)\}$.

Since the definition of continuity cannot be used in this context, we must use an indirect way to characterize the fact that a digital $\mathcal{R}-$path can be continuously deformed into another one. Thus, from Lemma 2.3, some convenient definition of homotopy may be given using small , but not too small, simply connected cells, called *elementary* $\mathcal{R}-$*deformation cells*, in which two paths with the same extremities are intuitively always homotopic. Given a digital space $\mathcal{E}$ and two complementary adjacency relations $(\mathcal{R}, \overline{\mathcal{R}})$, one can define the *elementary* $\mathcal{R}$*-deformation cells* in this space. Then a definition of *elementary* $\mathcal{R}-$*deformation* can be given as follows : two paths are said to be equivalent up to an elementary $\mathcal{R}-$deformation if they are almost the same expect in an elementary $\mathcal{R}-$deformation cell.

We give in the following the Definitions of the elementary deformation cells for classical digital spaces and adjacency relations. Note that the *choice* of these cells satisfies two main criterions. The first one is that any connected subset X of the cell must be simply connected according to Definition 2.6 applied to its continuous analogue. The second criterion is that the deformation cell must be large enough to allow elementary deformations to be performed inside thin parts of an object where it should be possible. The first criterion is illustrated by Figure 2.8 where we have depicted few $26-$connected subsets of a 2×2×2 cube in $\mathbb{Z}^3$. In these cases, the continuous analogue in $\mathbb{R}^3$ of these objects is nothing but the set of points of the voxels (as unit cubes of $\mathbb{R}^3$) of the object. It is clear that any of these sets is simply connected. In Figure 2.10(a) we have depicted a $6-$connected subset of a 2×2×2 cube. In this case, and since the adjacency used for the complement of the object is the 26−adjacency relation, it follows that the two white (i.e. not visible) voxels of the cube are 26−connected and the continuous analogue (Figure 2.10(b)) is not simply connected. Thus, the elementary 26−deformation cell cannot be a 2×2×2 cube; whereas a 2×2 square will be convenient since it will satisfy the latter criterion but also the second one (it is large enough). However, for $(n, \overline{n}) \in \{(26, 6), (18, 6), (6+, 18)\}$ a 2×2 square may not be sufficient to perform deformations of n−paths in an n−connected object. For example, no elementary not trivial deformation would be possible in a 2×2 square in the digital plane depicted in Figure 2.9. Thus, for $(n, \overline{n}) \in \{(26, 6), (18, 6), (6+, 18)\}$ we will define an elementary deformation cell as a 2×2×2 cube.

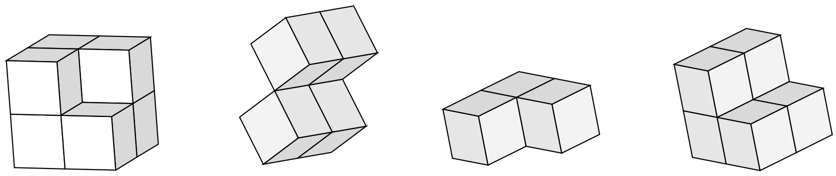


Figure 2.8: Few subsets of a 2×2×2 cube in $\mathbb{Z}^3$.

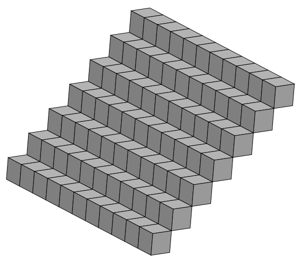
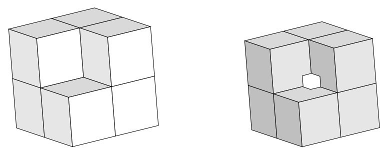


Figure 2.9: A 2×2 square is not a large enough deformation cell for $(n, \overline{n}) = (26, 6)$ or $(18, 6)$.



(a) (b)

Figure 2.10: A subset of a 2×2×2 cube for $(n, \overline{n}) = (6; 26)$.

In the sequel of this subsection, $\mathcal{E}$ is a digital space, the couple $(\mathcal{R}, \overline{\mathcal{R}})$ stands for two complementary adjacency relations and X is a subset of $\mathcal{E}$.

Definition 2.9 (elementary deformation cells in $\mathbb{Z}^2$ and $\mathbb{Z}^3$) *Depending on the adjacency relation couple $(n, \overline{n})$ we have :*

- *If $\mathcal{E} = \mathbb{Z}^2$, an elementary deformation cell is an 2×2 square of pixels.*
- *If $\mathcal{E} = \mathbb{Z}^3$ and $(n, \overline{n}) = (6, 26)$, an elementary deformation cell is a 2×2 square of voxels.*

- *If $\mathcal{E} = \mathbb{Z}^3$ and $(n, \overline{n}) \in \{(26, 6), (18, 6), (6+, 18)\}$, an elementary deformation cell is an 2×2×2 cube of voxels.*

The following remark will be of interest in the sequel.

Remark 2.2 *Two $\mathcal{R}-$adjacent spels are always included in some elementary $\mathcal{R}-$deformation cell associated to $\mathcal{E}$.*

We can now define the relation of elementary deformation between two digital paths an then the homotopy relation between digital paths.

Definition 2.10 (elementary $\mathcal{R}-$deformation) *Let π and π' be two $\mathcal{R}-$paths in X. We say that π and π'* are the same up to an elementary $\mathcal{R}-$deformation, *and we denote $\pi \sim_{\mathcal{R}} \pi'$ if π and π' are of the form $\pi = \pi_1.\gamma.\pi_2$ and $\pi' = \pi_1.\gamma'.\pi_2$ where γ and γ' are two $\mathcal{R}-$paths with the same extremities and both included in an* elementary $\mathcal{R}-$deformation cell *of $\mathcal{E}$.*

Definition 2.11 ($\mathcal{R}-$homotopy of digital paths) *Two $\mathcal{R}-$paths π and π' in X are are said to be* $\mathcal{R}-$homotopic with fixed extremities in X, *and we denote $\pi \simeq_{\mathcal{R}} \pi'$ if there exists a sequence $\pi_1, \ldots, \pi_l$ of $\mathcal{R}-$paths in X such that $\pi = \pi_1$, $\pi' = \pi_l$ and for $i = 1, \ldots, l-1$ the path π_i and π_{i+1} are the same up to an elementary $\mathcal{R}-$deformation.*

Definition 2.12 (reducible path) *A closed $\mathcal{R}-$path π from a spel x to x in X is said to be* $\mathcal{R}-$reducible in X *if $\pi \simeq_{\mathcal{R}} (x, x)$ in X.*

Remark 2.3 *In the following, we will omit the words "with fixed extremities" since the definition of the elementary $\mathcal{R}-$deformation implies that the deformation is achieved with fixed extremities.*

Two examples of elementary $n-$deformations are depicted by Figure 2.11 for $n = 4$ in $\mathbb{Z}^2$ and in Figure 2.12 for $n = 6+$ in $\mathbb{Z}^3$.

Then, we can give a definition of *simply connected digital objects* as follows. Note that we have used the notion of simply connectedness for subsets of deformation cells in order to obtain Definition 2.9. One could think that simply connectedness is thus not well defined. However, the fact that a very small subset of the space, like elementary deformations cells are, is simply connected, is admissible with no need of more mathematical tools than those

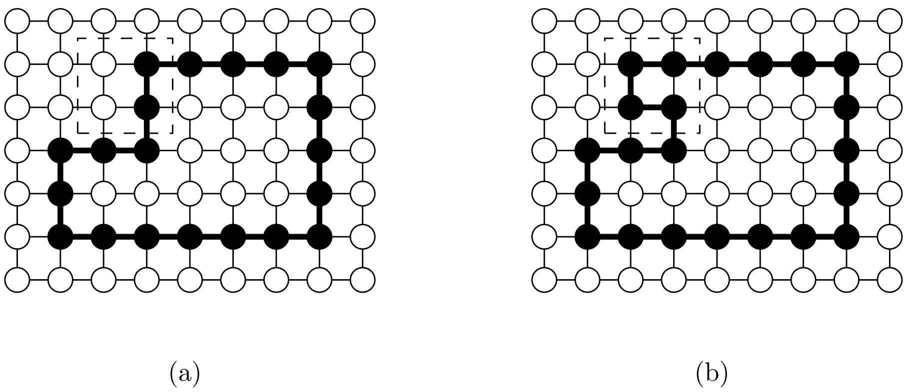


Figure 2.11: The two black 4−paths are the same up to an elementary 4−deformation.

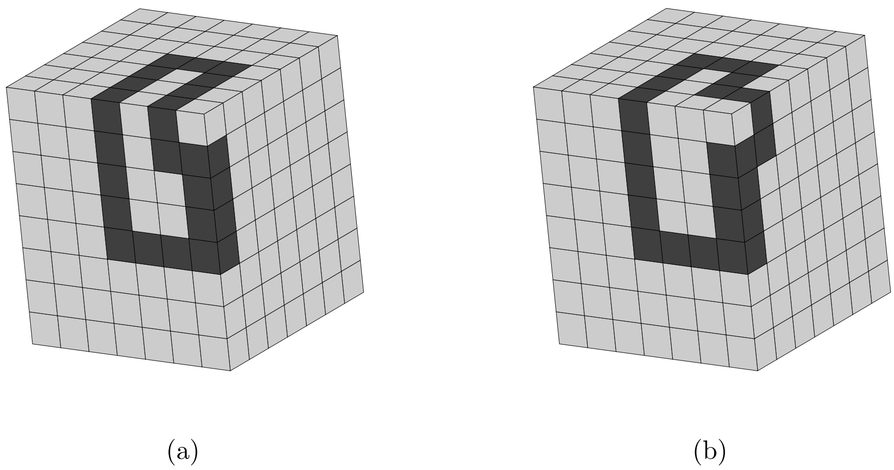


Figure 2.12: The two black (6+)−paths are the same up to an elementary deformation.

exposed here. This admitted property then leads to the commonly used deformation relations, which are finally no more than conventional ones.

Now, we can define *simply connected sets* with no need of a definition of the continuous analogue of a digital object. At this step, many definitions can be given which are similar to their continuous analogues but which are stated in an only discrete way.

Definition 2.13 (simply $\mathcal{R}-$connected set) *The set X is said to be* simply $\mathcal{R}-$connected *if for any base point $x \in X$, every closed $\mathcal{R}-$path π from x to x in X is $\mathcal{R}-$reducible in X.*

An example of a subset of $\mathbb{Z}^3$ which is not simply connected is depicted in Figure 2.13.

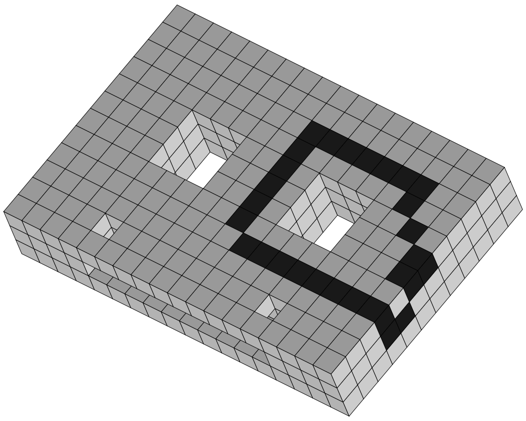


Figure 2.13: This object of $\mathbb{Z}^3$ is not simply 18−connected since the 18−path (in black) is not 18−reducible.

The following lemma will be useful in the next subsection.

Lemma 2.4 *Let π be a $\mathcal{R}-$path from a spel y_0 to a spel y_p in an object X. Then, the closed $\mathcal{R}-$path $\pi.\pi^{-1}$ from y_0 to y_0 is $\mathcal{R}-$homotopic to the path (y_0, y_0) in X.*

Proof : Let $\pi = (y_0, \dots, y_p)$. Then, for $k \in \{0, \dots, p\}$ be denote by β^k the $\mathcal{R}-$path $(y_0, y_1, \dots, y_k)$. We first prove that for all $j \in \{1, \dots, p\}$ the closed path $\beta^j.(\beta^j)^{-1}$ is $\mathcal{R}-$homotopic to the path $\beta^{j-1}.(\beta^{j-1})^{-1}$. Indeed, for such j we have $\beta^j.(\beta^j)^{-1} = \beta^{j-1}.(y_{j-1}, y_j, y_{j-1}).(\beta^{j-1})^{-1}$. Now, since the two $\mathcal{R}-$adjacent spels y_{j-1} and y_j belong to a deformation cell (Remark 2.2), it follows that the path $\beta^j.(\beta^j)^{-1}$ is $\mathcal{R}-$homotopic to the path $\beta^{j-1}.(y_{j-1}).(\beta^{j-1})^{-1} = \beta^{j-1}.(\beta^{j-1})^{-1}$. Finally, we obtain that $\pi.\pi^{-1} = \beta^p.(\beta^p)^{-1} \simeq_{\mathcal{R}} \beta^{p-1}.(\beta^{p-1})^{-1} \simeq_{\mathcal{R}} \dots \simeq_{\mathcal{R}} \beta^0.(\beta^0)^{-1} = (y_0, y_0)$. $\square$

Using Lemma 2.4, we can already prove the following Proposition which will be useful in the sequel.

Proposition 2.5 *The set X is simply $\mathcal{R}-$connected if and only if any two $\mathcal{R}-$paths π and π' with same extremities are $\mathcal{R}-$homotopic in X.*

Proof : First, we suppose that any two paths π and π' are $\mathcal{R}-$homotopic with fixed extremities in X. Then, let x be a spel of X and $c = (x_0, \dots, x_q)$ be a closed $\mathcal{R}-$path from x to x in X (i.e. $x_0 = x_q = x$). It follows that $c \simeq_{\mathcal{R}} (x, x)$ so c is reducible in X.

Now, suppose that any closed path in X is reducible in X. Then, let π_1 and π_2 be two $\mathcal{R}-$paths from x to y in X. We obviously have $(y, y) \simeq_{\mathcal{R}} \pi_1^{-1}.\pi_2$ in X since $\pi_1^{-1}.\pi_2$ is a closed $\mathcal{R}-$path in X. Since π_1 is obviously $\mathcal{R}-$homotopic to the path $\pi_1.(y, y)$ it follows that $\pi_1 \simeq_{\mathcal{R}} \pi_1.\pi_1^{-1}.\pi_2$ in X. From Lemma 2.4, we have $\pi_1.\pi_1^{-1} \simeq_{\mathcal{R}} (x, x)$ and finally $\pi_1 \simeq_{\mathcal{R}} (x, x).\pi_2 \simeq_{\mathcal{R}} \pi_2$ in X. $\Box$

The following lemma will be useful in the sequel.

Lemma 2.6 *Let π be a closed $\mathcal{R}-$path in X which is reducible in X and let π' be a closed $\mathcal{R}-$path such that π and π' are the same up to a change of parameter. Then π' is also reducible.*

Proof : If the two closed $\mathcal{R}-$paths $\pi = (y_0, \dots, y_p)$ and $\pi' = (y'_0, \dots, y'_p)$ are the same up to a change of parameter, then one of the two following situation occurs :

i) $\pi' = (y_{k_0}, y_{k_0+1}, \dots, y_p).(y_0, y_1, \dots, y_{k_0})$

ii) $\pi' = (y_{k_0}, y_{k_0-1}, \dots, y_0).(y_p, y_{p-1}, \dots, y_{k_0})$

Case *i)* : Let γ be the path $(y_0, \dots, y_{k_0})$. Then, from Lemma 2.4, the path $\gamma^{-1}.\gamma$ which is a closed path from y_{k_0} to y_{k_0} is reducible so that π' is $\mathcal{R}-$homotopic to the path $\gamma^{-1}.\gamma.\pi' = \gamma^{-1}.\gamma.(y_{k_0}, y_{k_0+1}, \dots, y_p).(y_0, y_1, \dots, y_{k_0})$. where $\gamma.(y_{k_0}, y_{k_0+1}, \dots, y_p)$ is nothing but the path π. Then, it is straightforward that $\gamma^{-1}.\gamma.\pi'$ is $\mathcal{R}-$homotopic to the path $\gamma^{-1}.(y_0, y_1, \dots, y_{k_0}) = \gamma^{-1}.\gamma$ itself reducible according to Lemma 2.4. Finally, π' is reducible in X.

Case *ii)* : The path π' of case *ii)* is nothing but the invert of the path π' of case *i)*. Then, we should accept without proof that if a path c is reducible, so is the path c^{-1}. Then, if the path π' of case *i)* is reducible, so is the path π' of case *ii)*. $\Box$

2.3 A digital fundamental group

In this section, we introduce an important tool of the digital framework which will provide a formal and rigorous way to characterize topology preservation within digital spaces : the *digital fundamental group* which was first introduced by T.Y. Kong [45]. Note that one of the purposes of Part II is to state a new theorem which shows that this tool is sufficient to characterize *lower homotopy* within the digital space constituted by a digital surface (the notion of lower homotopy will be formally defined in Chapter 3). On the other hand, the fundamental group is also used in the characterization of 3D simple points which was simplified in Part III. We recall that the digital fundamental group is in fact the central notion in our study.

2.3.1 Definition

The digital fundamental group has been first defined in [45]. It's definition is similar to the definition of the classical fundamental group in algebraic topology, and relies on the notion of homotopy (see [98]).

Let X be an $\mathcal{R}-$connected subset of $\mathcal{E}$ and $B \in X$ be a spel which is called the *base point* (or *base spel*). We denote by $A^B_{\mathcal{R}}(X)$ the set of all the closed $\mathcal{R}-$paths from B to B in X. First, we observe that the $\mathcal{R}-$homotopy relation is an equivalence relation for paths of $A^B_{\mathcal{R}}(X)$.

Let $\Pi^{\mathcal{R}}_1(X, B)$ be the set of equivalence classes of paths of $A^B_{\mathcal{R}}(X)$ under the $\mathcal{R}-$homotopy equivalence relation (i.e. $\Pi^{\mathcal{R}}_1(X, B) = A^B_{\mathcal{R}}(X)_{/ \simeq_{\mathcal{R}}}$).

Notation 2.1 *For any closed path $c \in A^B_{\mathcal{R}}(X)$, we denote by $[c]_{\Pi^{\mathcal{R}}_1(X,B)}$ the equivalence class of the path c in $\Pi^{\mathcal{R}}_1(X, B)$. When no confusing is possible, we will denote it briefly by $[c]$. Note that the former notation is slightly different from the one given in Definition 1.4. Furthermore, in order to improve readability, we may denote $[1]_{\Pi^{\mathcal{R}}_1(X,B)}$ for $[(B, B)]_{\Pi^{\mathcal{R}}_1(X,B)}$.*

Now, we can define the subset C of $(\Pi^{\mathcal{R}}_1(X, B) \times \Pi^{\mathcal{R}}_1(X, B)) \times \Pi^{\mathcal{R}}_1(X, B)$:

Definition 2.14 (C) *We denote by C the subset of $(\Pi^{\mathcal{R}}_1(X, B) \times \Pi^{\mathcal{R}}_1(X, B)) \times \Pi^{\mathcal{R}}_1(X, B)$ defined by : $C = \{\ ([a], [b]), [a.b]) \mid a, b \in A^B_{\mathcal{R}}(X)\ \}$.*

We will need the following lemma in order to prove that C is the graph of a map (Proposition 2.8).

Lemma 2.7 ($\simeq_{\mathcal{R}}$ is compatible with .) *Let a, b, a' and b' be four paths of $A^B_{\mathcal{R}}(X)$, if $a \simeq_{\mathcal{R}} a'$ and $b \simeq_{\mathcal{R}} b'$, then $a.b \simeq_{\mathcal{R}} a'.b'$.*

Proof : From the very definition of $\mathcal{R}-$homotopy, if $a \simeq_{\mathcal{R}} a'$ then $a.b \simeq_{\mathcal{R}} a'.b$. Indeed, if $a = a_0 \sim_{\mathcal{R}} \sim_{\mathcal{R}} a_1 \ldots \sim_{\mathcal{R}} a_l = a'$ is a sequence of elementary $\mathcal{R}-$deformations in X, then so is $a.b = a_0.b \sim_{\mathcal{R}} \sim_{\mathcal{R}} a_1.b \ldots \sim_{\mathcal{R}} a_l.b = a'.b$. Similarly, $a'.b \simeq_{\mathcal{R}} a'.b'$. $\square$

Proposition 2.8 *C is the graph of a map.*

Proof : Let $(([a],[b]),[a.b]) \in C$ and $(([a'],[b']),[a'.b']) \in C$. Suppose that $([a],[b]) = ([a'],[b'])$, i.e. $a \simeq_{\mathcal{R}} a'$ and $b \simeq_{\mathcal{R}} b'$. From Lemma 2.7, it follows that $a.b \simeq_{\mathcal{R}} a'.b'$, i.e. $[a.b] = [a'.b']$. $\square$

Definition 2.15 (operation $*$) *From Proposition 2.8 the following map is well defined :*

$$\begin{array}{rrcl} * \; : & \Pi_1^{\mathcal{R}}(X,B) \times \Pi_1^{\mathcal{R}}(X,B) & \longrightarrow & \Pi_1^{\mathcal{R}}(X,B) \\ & ([\pi_1],[\pi_2]) & \longmapsto & [\pi_1] * [\pi_2] = [\pi_1.\pi_2] \end{array}$$

Then, $*$ is an internal operation on $\Pi_1^{\mathcal{R}}(X,B)$.

Proposition 2.9 *$(\Pi_1^n(X,B),*)$ is a group.*

Proof : Let π_{id} be the trivial path (B,B).

- The operation $*$ is associative, indeed, from the very definitions of "$*$" and ".", $([a]*[b])*[c] = [a.b]*[c] = [(a.b).c] = [a.(b.c)] = [a]*[b.c] = [a]*([b]*[c])$.

- For any class $[\pi] \in \Pi_1^n(X,B)$ we have $[\pi]*[\pi_{id}] = [\pi_{id}]*[\pi] = [\pi]$. Indeed, for any path $\pi \in A^B_{\mathcal{R}}(X)$ the paths $\pi.(B,B)$ and $(B,B).\pi$ are both $\mathcal{R}-$homotopic to π (in fact $\pi.(B,B)$ and π are obviously the same up to an elementary $\mathcal{R}-$deformation). Thus, π_{id} is the identity element of the structure $(\Pi_1^{\mathcal{R}}(X,B),*)$.

- For any equivalence class $[\pi]$ in $\Pi_1^{\mathcal{R}}(X,B)$, we define $[\pi]^{-1} = [\pi^{-1}]$. Then, $[\pi]*[\pi]^{-1} = [\pi]*[\pi^{-1}] = [\pi.\pi^{-1}]$. From Lemma 2.4, the path $\pi.\pi^{-1}$ is $\mathcal{R}-$homotopic to the trivial path (B,B). Thus, $[\pi]*[\pi]^{-1} = [\pi_{id}]$. Similarly, $[\pi]^{-1}*[\pi] = [\pi_{id}]$.

□

The following proposition means that the digital fundamental group, up to an isomorphism, *does not depend* on the choice of a base point in a same $\mathcal{R}-$connected component of X.

Proposition 2.10 *Let B and B' be two spels which are $\mathcal{R}-$connected in X. Then $\Pi_1^{\mathcal{R}}(X,B)$ and $\Pi_1^{\mathcal{R}}(X,B')$ are isomorphic.*

Proof : Let c be a $\mathcal{R}-$path from B to B' in X and I be the following map :

$$\begin{array}{rcccl} I & : & \Pi_1^{\mathcal{R}}(X,B) & \longrightarrow & \Pi_1^{\mathcal{R}}(X,B') \\ & & [\pi] & \longmapsto & [c^{-1}.\pi.c] \end{array}$$

First, we prove that I is a group morphism. Let π_1 and π_2 be two paths in $A_{\mathcal{R}}^B(X)$. We have $I([\pi_1]*[\pi_2]) = I([\pi_1.\pi_2]) = [c^{-1}.\pi_1.\pi_2.c]$. On the other hand, $I([\pi_1]) = [c^{-1}.\pi_1.c]$ and $I([\pi_2]) = [c^{-1}.\pi_2.c]$ so $I([\pi_1])*I([\pi_2]) = [c^{-1}.\pi_1.c]*[c^{-1}.\pi_2.c] = [c^{-1}.\pi_1.c.c^{-1}.\pi_2.c]$. But from Lemma 2.4, $c.c^{-1} \simeq_{\mathcal{R}} (B,B)$ so that $[c^{-1}.\pi_1.c.c^{-1}.\pi_2.c] = [c^{-1}.\pi_1.\pi_2.c]$. Finally, $I([\pi_1]*[\pi_2]) = I([\pi_1])*I([\pi_2])$.

Now, let $\pi' \in A_{\mathcal{R}}^{B'}(X)$ (i.e. $[\pi'] \in \Pi_1^{\mathcal{R}}(X,B')$), then the path $\pi = c.\pi'.c^{-1}$ from B to B in X is such that $I([\pi]) = [\pi']$. Indeed, $I([c.\pi'.c^{-1}]) = [c^{-1}.c.\pi'.c^{-1}.c]$ and from Lemma 2.4, we have $c^{-1}.c.\pi'.c^{-1}.c \simeq_{\mathcal{R}} \pi'$ so $[c^{-1}.c.\pi'.c^{-1}.c] = [\pi']$. Finally, I is onto.

Suppose that π_1 and π_2 are two paths of $A_{\mathcal{R}}^B(X)$ such that $I([\pi_1]) = I([\pi_2])$. It follows that the paths $c^{-1}.\pi_1.c$ and $c^{-1}.\pi_2.c$ are $\mathcal{R}-$homotopic in X. Now, if $c^{-1}.\pi_1.c \simeq_{\mathcal{R}} c^{-1}.\pi_2.c$ then $c.c^{-1}.\pi_1.c \simeq_{\mathcal{R}} c.c^{-1}.\pi_2.c$ (observe that the latter two paths are paths from B to B'). Similarly, we also deduce that $c.c^{-1}.\pi_1.c.c^{-1} \simeq_{\mathcal{R}} c.c^{-1}.\pi_2.c.c^{-1}$ (paths form B to B). From Lemma 2.4, it follows that $\pi_1 \simeq_{\mathcal{R}} \pi_2$ so that $[\pi_1] = [\pi_2]$. Finally, I is one to one. □

2.3.2 Group morphism induced by the inclusion map

The notion of a group morphism induced by an inclusion map will be used in next Parts to characterize topology preservation when a set of spels (Part II) or a single spel (Part III) is removed from an object.

Let $Y \subset X$ be two subsets of a digital space $(\mathcal{E},\mathcal{R})$ and B be a spel of Y. We denote by $i:\ Y \longrightarrow X$ the inclusion map of Y in X (i.e. $\forall x \in Y$, $i(x) = x$). Then, one can define a map i_+ from $A_{\mathcal{R}}^B(Y)$ to $A_{\mathcal{R}}^B(X)$ as follows :

$$\begin{array}{rrcl} i_+ \ : & A^B_{\mathcal{R}}(Y) & \longrightarrow & A^B_{\mathcal{R}}(X) \\ & (x_0, x_1, \ldots, x_q) & \longmapsto & (i(x_0), i(x_1), \ldots, i(x_q)) = (x_0, x_1, \ldots, x_q) \end{array}$$

This map simply send a closed path of $A^B_{\mathcal{R}}(Y)$ to the same closed path as a path of $A^B_{\mathcal{R}}(X)$. Then, we can define the canonical following group morphism :

Definition 2.16 *We define i_*, the* group morphism induced by the inclusion of Y in X *by :*

$$\begin{array}{rrcl} i_* \ : & \Pi_1^{\mathcal{R}}(Y, B) & \longrightarrow & \Pi_1^{\mathcal{R}}(X, B) \\ & [c]_{\Pi_1^{\mathcal{R}}(Y,B)} & \longmapsto & [c]_{\Pi_1^{\mathcal{R}}(X,B)} \end{array}$$

This map is well defined since two $\mathcal{R}-$paths which are $\mathcal{R}-$homotopic in Y are obviously $\mathcal{R}-$homotopic in X so that the an homotopy class $[c]_{\Pi_1^{\mathcal{R}}(Y,B)}$ is sent by this map on a single homotopy class in $\Pi_1^{\mathcal{R}}(X, B)$. Furthermore, this map is clearly a group morphism from the very definition of the operation $*$ (Definition 2.15).

Chapter 3

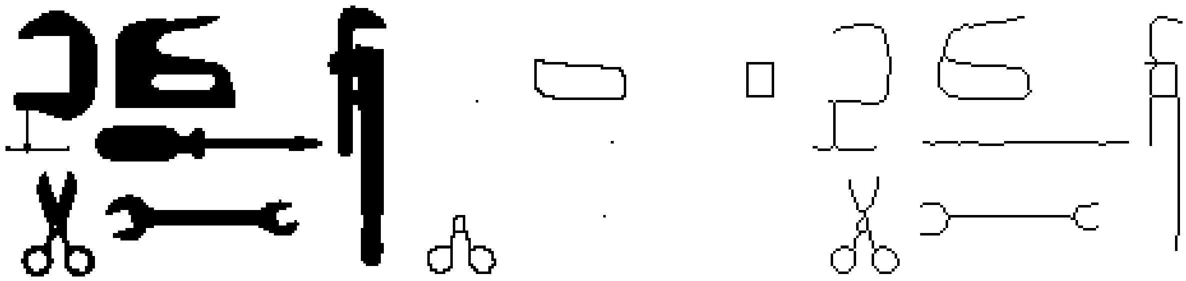
Topology preservation

In this chapter, we will introduce the notion of topology preserving deformations within digital objects. While Sections 2.2 and 2.3 were dedicated to the definition of some *continuous* deformations for digital paths, another notion of homotopy can be defined for digital objects themselves. First, we will recall an important application of this notion : the so called homotopic thinning algorithms. Then, we will introduce several generic definitions before giving their illustrations in the digital space $\mathbb{Z}^2$.

3.1 Topology preserving deformations

Here we give a first approach to the notion *topology preservation in a digital space*, taking examples in the digital space $\mathbb{Z}^2$ with no formal definition of this central notion. However, this section will help to motivate the few definitions which will be given in Section 3.2. Then, in a following section we will use these definitions to formalize the main results obtained by several authors about topology preservation in $\mathbb{Z}^2$.

In the image analysis chain, and for example for pattern recognition purpose, it is sometimes necessary to extract some topological informations from images. Then, this can be achieved after a first step which consists in the use of an homotopic thinning algorithm. Here, *homotopic* means that such an algorithm is expected not to change the topological properties of the given image. However, given an object X in a digital space $\mathcal{E}$, it produces an object $Y \subset X$ which is topologically equivalent to the object X but which has a lower size (according to the number of spels it contains). More precisely, Y is one dimension less than X as illustrated in Figure 3.1(b), where every large areas with holes are reduced to a network made of curves (see scissors of Figure 3.1(a)), and every parts with no hole are reduced to isolated pixels. In practice, it is often useful to use some thinning algorithms which also preserve some geometrical informations of the objects as depicted in Figure 3.1(c). Now, the most commonly used method to operate such an extraction consists in a sequential, and sometimes parallel, deletion of spels, taking care not to change the topological properties of the image. In this case, like in the case of continuous deformations which will be described next, a characterization of a topology preservation is needed for an operation which consists in the removal of a spel.



(a) The initial image.

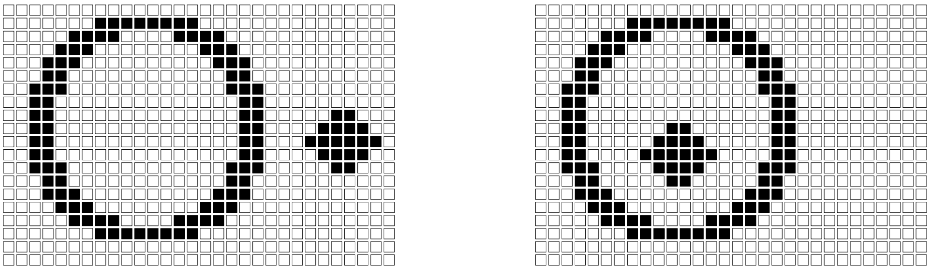
(b) Resulting image after a complete thinning of the initial image.

(c) Resulting image after a partial thinning algorithm.

Figure 3.1: Illustration of the possible result of an homotopic thinning algorithm in $\mathbb{Z}^2$.

On the other hand, it is sometimes useful to simply check if some kind of continuous deformations can be applied to an image in order to obtain a second one. As an example, one may wonder if the image of Figure 3.2(a) can be continuously deformed into the one of Figure 3.2(b). Obviously, the definition of a *continuous* deformation in a digital space is expected not to allow such kind of deformation. In this case, the notions which will follow are also of interest.

Indeed, a classical way to define a topology preserving deformation of a digital object is to break it up in a sequence of elementary deformations each of which consists in the deletion or addition of a single spel of the object. The fundamental notion of this method is the notion of *simple spel*. A *simple spel* is thus defined as a spel the deletion of which leaves the topological properties of an object unchanged. However, this definition is clearly not acceptable. Indeed, no precise definition has been given yet of the topological properties of an object. In fact, this latter definition depends on the considered digital



(a) (b)

Figure 3.2: Can the two images be continuously deformed one into each other ?

space. Clearly, an object in $\mathbb{Z}^2$ is characterized by its number of black and white connected components and by the surrounding relations between these components (see Figure 3.2). This definition leads to an immediate definition of a simple pixel which itself leads to a *local characterization* of such pixels.

In next section, we will introduce the notions of simpleness property, local characterization, lower homotopy and more. Note that these definitions do not depend on the digital space considered but only on the definition of a first property : simpleness.

3.2 Simpleness property and homotopy

Since the definition of a *simple spel* is very dependent on the digital space considered, and since many further definitions use this latter notion, we first introduce the notion of a *simpleness property* as a generic one. However, since this property is strictly related to topology preservation for our purpose, we choose to use the word "simpleness" in order to recall that this kind of property must be precisely defined for each kind of digital space. Moreover, in order to give a practical meaning of these definitions, and as a first introduction to the questions which will be investigated in Parts II and III, we will also illustrate these notions in the case of the digital space $\mathbb{Z}^2$ in next subsection.

Definition 3.1 (simpleness property) *Let* $(\mathcal{E}, \mathcal{R})$ *be a digital space and* $P(\mathcal{E})$ *be the set of subsets of* $\mathcal{E}$. *We define a* simpleness property $\mathcal{S}$ in $\mathcal{E}$ *as a map from* $P(\mathcal{E}) \times \mathcal{E}$ *to* $\{0, 1\}$. *If* X *is a subset of* $\mathcal{E}$, *a spel* x *is called* $\mathcal{S}-$simple for X *if* $\mathcal{S}(X, x) = 1$.

Following the definition of an $\mathcal{S}-$simple spel, we can define $\mathcal{S}-$simple sets which are sets one can order as a sequence such that an iterative deletion of spels following this sequence only removes $\mathcal{S}-$simple spels.

Definition 3.2 ($\mathcal{S}-$simple set) *Let $(\mathcal{E},\mathcal{R})$ be a digital space and $\mathcal{S}$ be a simpleness property on $\mathcal{E}$. Furthermore, let $X subset \mathcal{E}$ and let $D = \{x_0, x_1, \ldots, x_l\}$ be a subset of X. The set D is called an* $\mathcal{S}-$simple set *for X if there exists a permutation σ of $\{0,\ldots,l\}$ such that for $i = 0,\ldots,l$ the spel $x_{\sigma(i)}$ is $\mathcal{S}-$simple for $X \setminus \{x_{\sigma(j)} \mid j < i\ \}$.*

Now, the definition of simple spels allows us to define one of the fundamental notion of digital topology especially in the context of thinning algorithms : the *lower homotopy relation.*

Definition 3.3 (lower $\mathcal{S}-$homotopy) *Let $Y \subset X$ be two subsets of a digital space $\mathcal{E}$ and let $\mathcal{S}$ be a simpleness property in $\mathcal{E}$. The set Y is said to be* lower $\mathcal{S}-$homotopic to *X if the set $Y \setminus X$ is an $\mathcal{S}-$simple set for X. In other words, there exists a sequence $S_0 \subset S_1 \subset \ldots \subset S_k$ of subsets of $\mathcal{E}$ such that $Y = S_0$, $X = S_k$ and for $i = 0 \ldots k-1$, the set S_i is obtained by deletion of an $\mathcal{S}-$simple spel in S_{i+1}.*

If the definition of lower homotopy is clearly related to the homotopic thinning process, it is however useful to define a more general relation which links digital objects which can be said equivalent up a to some continuous deformation. Indeed, for image analysis purpose, one may want to characterize the fact that any of the objects depicted in Figure 3.3 can be obtained by a continuous deformation of each other. A natural way to define such an equivalence relation between objects is to use the notion of simple spel. This kind of deformation is sometimes called a *simple deformation* as in [91]. Here we will use the term of *symmetric homotopy.*

Figure 3.3: Two subsets of $\mathbb{Z}^2$ which are the same up to a "continuous" deformation.

Definition 3.4 (symmetric $\mathcal{S}-$homotopy) *Let X and Y be two subsets of a digital space $\mathcal{E}$ and let $\mathcal{S}$ be a simpleness property on $\mathcal{E}$. The set X is said to be* symmetrically $\mathcal{S}-$homotopic to Y *if there exists a sequence $S_0, S_1, \ldots, S_k$ of subsets of $\mathcal{E}$ such that $X = S_0$, $Y = S_k$ and for $i = 0 \ldots k-1$, the set S_i is obtained by deletion of an $\mathcal{S}-$simple spel in S_{i+1} or the set S_{i+1} is obtained by deletion of an $\mathcal{S}-$simple spel in S_i.*

Now, we introduce the notions of a *local characterization* and a *local property.* A *local* characterization can be defined as a map from $P(\mathcal{E}) \times \mathcal{E}$ to $\{0,1\}$, the value of which is computed for each $x \in \mathcal{E}$ by considering a small set of spels.

Definition 3.5 (local characterization, local property) *Let $(\mathcal{E}, \mathcal{R})$ be a digital space and X be an object of $\mathcal{E}$. A* local characterization *is a map C of the form :*

$$\begin{array}{rcl} C \ : \ P(\mathcal{E}) \times \mathcal{E} & \longrightarrow & \{0,1\} \\ (X,x) & \longmapsto & \mathcal{V}(N_{\mathcal{R}}(x) \cap X) \end{array}$$

Where $\mathcal{V}$ is a map from $P(\mathcal{E})$ to $\{0,1\}$. We say that a spel $x \in \mathcal{E}$ satisfies the local characterization C in X if $C(X,x) = 1$. A property for spels and sets in a digital space is said to be local *if there exists a local characterization C such that a spel x satisfies the property with respect to X if and only if $C(X,x) = 1$.*

3.3 Topology preservation is $\mathbb{Z}^2$

In this section, X is a digital object in the digital space $\mathcal{E} = \mathbb{Z}^2$ and $(n, \overline{n}) \in \{(4,8),(8,4)\}$. It is commonly admitted that the deletion of a pixel x in $X \subset \mathbb{Z}^2$ changes the topology of X if it modifies the number of $n-$connected components of X or the number of $\overline{n}-$connected components of $\overline{X}$. This is summed up by the following definition of the simpleness property $\mathcal{S}_n$:

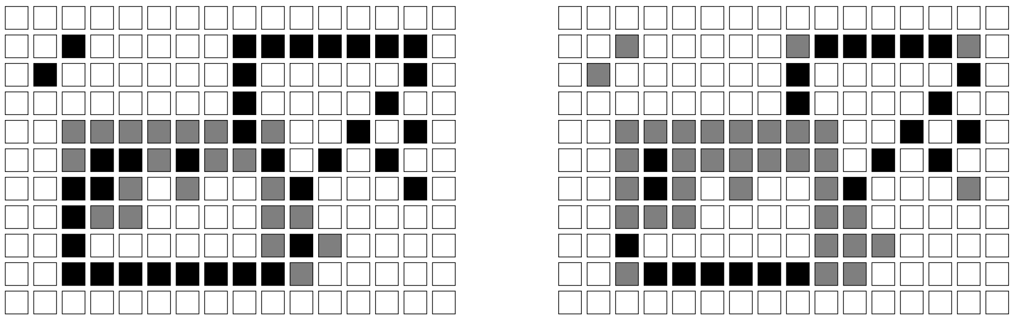
Definition 3.6 (simple pixel) *A pixel $x \in X$ is called $\mathcal{S}_n-$simple for X if the two following properties are satisfied :*

i) X and $X \setminus \{x\}$ have the same number of $n-$connected components.

ii) $\overline{X}$ and $\overline{X} \cup \{x\}$ have the same number of $\overline{n}-$connected components.

In order to avoid heavy notations and since no confusion is possible, we will abbreviate S_n-simple by $n-$simple.

We may illustrate on an example that deletion or addition of pixels which are $n-$simple cannot change the topological properties of a two dimensional digital image. Indeed, as previously mentioned, such an image is characterized by its number of black and white connected components but also by their surrounding relations. In Figure 3.4, we have depicted for an object X the set of pixels (in grey) which satisfy the properties of Definition 3.6. In other words, one can remove any one of the grey pixels in such a way that the topological properties of the image are unchanged. Conversely, the removal of any black pixel does change the topological properties of the image (according to the adjacency relations couple $(n, \overline{n})$ which is considered).



(a) $(n, \overline{n}) = (4, 8)$ (b) $(n, \overline{n}) = (8, 4)$

$X = \{$ ■,■ $\}$

Figure 3.4: Illustration of the simpleness property $\mathcal{S}_n$ for $n \in \{4, 8\}$, only grey pixels are $n-$simple for X.

Now, a very useful property is that the simpleness property $\mathcal{S}_n$ can be locally characterized. Indeed, the following theorem can be found in [87] :

Theorem 5 (local characterization of simple pixels) *The pixel $x \in X$ is $n-$simple for X if and only if :*

i) $N_n(x) \cap X$ *is not empty and $n-$connected.*

ii) $N_{\overline{n}}(x) \cap \overline{X}$ *is not empty and $\overline{n}-$connected.*

It is clear that this latter theorem provides a local characterization of the simpleness property $\mathcal{S}_n$ in the sense of Definition 3.5 since $N_n(x)$ and $N_{\overline{n}}(x)$ are both subsets of $N_8(x)$. In Figure 3.5, we have depicted a few examples of $8-$neighborhoods and given their corresponding simpleness property according to the local characterization. For example, the black pixel x of Figure 3.5(f) is not 8−simple since $N_4(x) \cap \overline{X} = \emptyset$. At this step, we observe that the problem of topology preservation for a digital image in $\mathbb{Z}^2$ is reduced to connectivity consideration in the neighborhood of a pixel. In next Parts, we will see that the same remark holds for some other kinds of digital spaces.

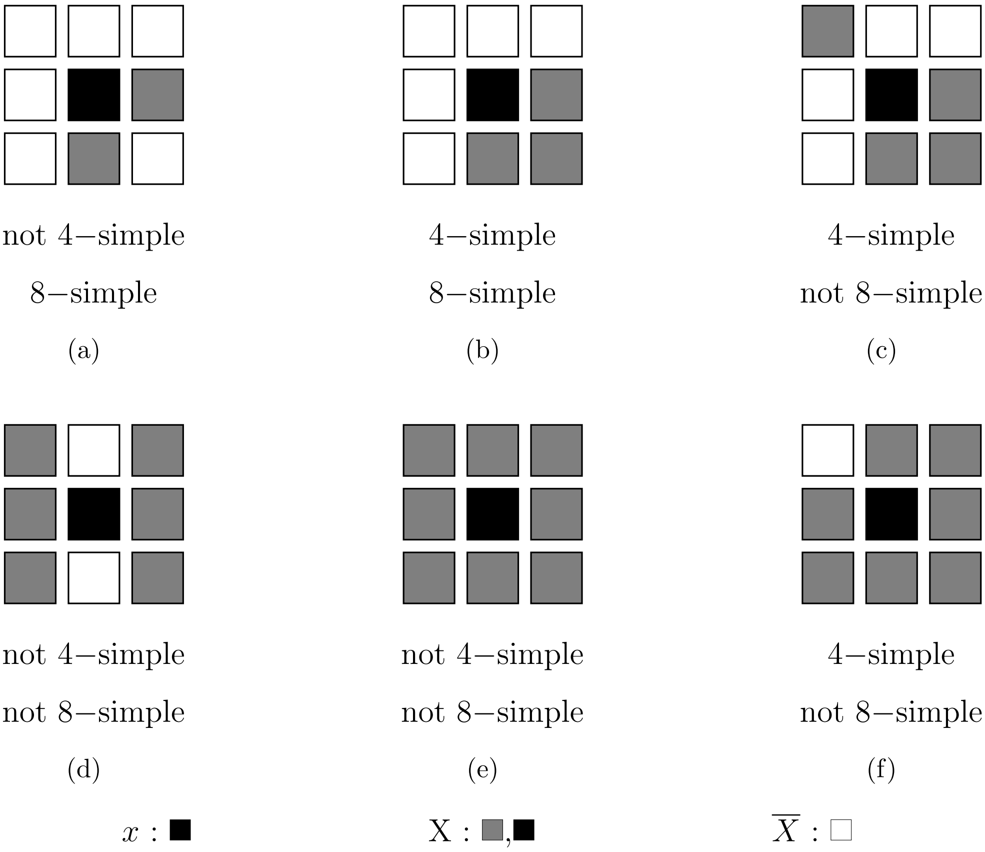


Figure 3.5: Simple and not simple pixels.

The definition of $n-$simple pixels in $\mathbb{Z}^2$ leads to the definitions of $n-$simple sets, lower $n-$homotopy and symmetric $n-$homotopy following Definitions 3.2, 3.3 and 3.4.

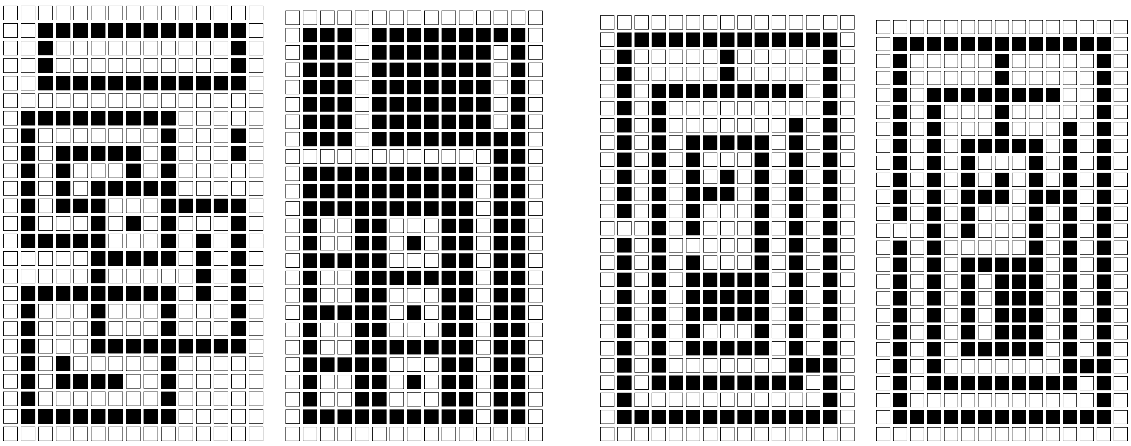
Now, an important result is that it is possible to characterize the fact that a subset Y of X is lower $n-$homotopic to X. Indeed, the following Proposition, which belongs to the folklore, leads to an efficiently computable algorithm to decide whether an object Y is lower $n-$homotopic or not to another object X in which it is included.

Proposition 3.1 (characterization of lower homotopy in $\mathbb{Z}^2$) *Let Y and X be two subsets of $\mathbb{Z}^2$ such that $Y \subset X$. The set Y is lower $n-$homotopic to X if and only if the two following conditions are satisfied :*

i) Each $n-$connected component of X contains a single $n-$connected component of Y.

ii) Each $\overline{n}-$*connected component of* $\overline{Y}$ *contains exactly one* $\overline{n}-$*connected component of* $\overline{X}$.

Now, a similar question can be raised for symmetric homotopy. Thus, a recent work of A. Rosenfeld and al. in [91] provides a characterization of symmetric homotopy which uses the notion of adjacency trees associated to a digital image. In other words, they gives a way to check that the objects depicted in Figure 3.6(a) are symmetrically $n-$homotopic whereas the two ones of Figure 3.6(b) are not. A similar use of adjacency trees in order to characterize symmetric homotopy can also be found in Bykov and Zerlakov [17].



(a) Two symmetrically homotopic objects.

(b) Two objects which are not symmetrically homotopic.

Figure 3.6: Symmetric homotopy in $\mathbb{Z}^2$.

First, we must recall the two following definitions which are slightly adapted from [91] :

Definition 3.7 (surrounding relation [91]) *Let* U, V *and* W *be three pairwise disjoint subsets of* $\mathbb{Z}^2$. *We say that* V $n-$separates U *from* W *if any* $n-$*path from a pixel of* U *to a pixel of* W *must contain a pixel of* V. *Now, let* B *be an* $n-$*connected component of a digital image* $\mathcal{I}$ *in* $\mathbb{Z}^2$ *and let* W *be an* $\overline{n}-$*connected white component of* $\mathcal{I}$. *We say that* B surrounds W *if* B $n-$*separates* W *from all but finitely many other pixels of* $\overline{B}$. *Respectively, we say that* W *surrounds* B *if* W $\overline{n}-$*separates* B *from all but finitely many other pixels of* $\overline{B}$.

Definition 3.8 (adjacency tree [91]) *Let* X *be an object in a digital image* $\mathcal{I}$ *of* $\mathbb{Z}^2$. *Then the graph, whose vertices are the black and white connected components of* $\mathcal{I}$ *and* *whose edges follow the symmetric closure of the surroundness relation between these connected components, is a rooted tree which is called the* adjacency tree of $\mathcal{I}$ *and is denoted by* $\mathcal{A}(\mathcal{I})$*.*

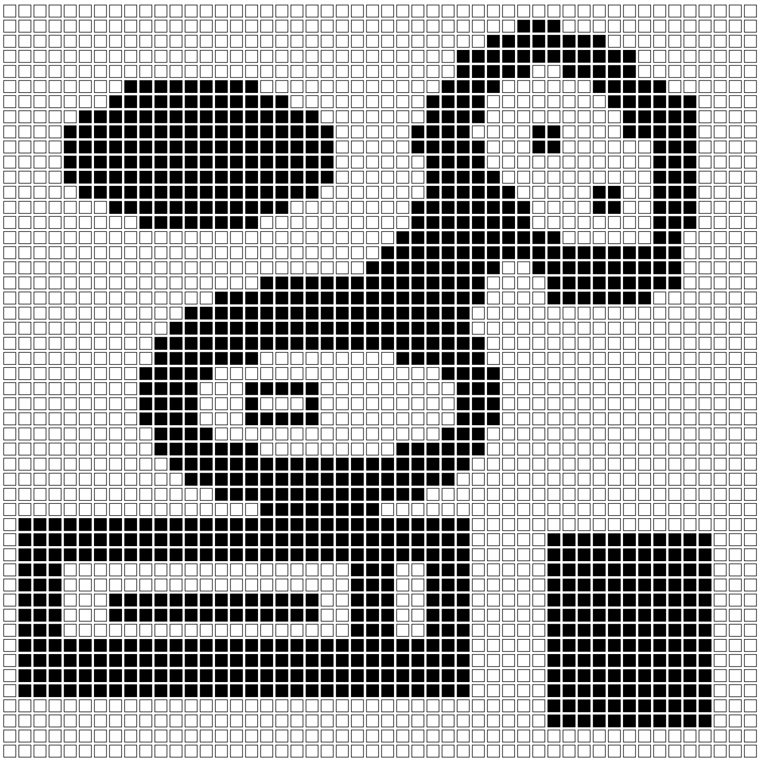
The leaves of such a tree are connected components which do not surround another component, like for example black components with no holes. In Figure 3.7(b) we have depicted the adjacency tree associated to the image shown in Figure 3.7(a). The root of this tree is the not finite white component which surrounds any other connected black or white connected component. In Figure 3.7(c), we give the canonical form of the adjacency tree where components are depicted with small circles with the corresponding color.

In [91], Rosenfeld and al. proved the following theorem which provides an algorithm to decide whether two 2D objects are symmetrically homotopic or not. Indeed, it reduces this problem to the graph isomorphism one.

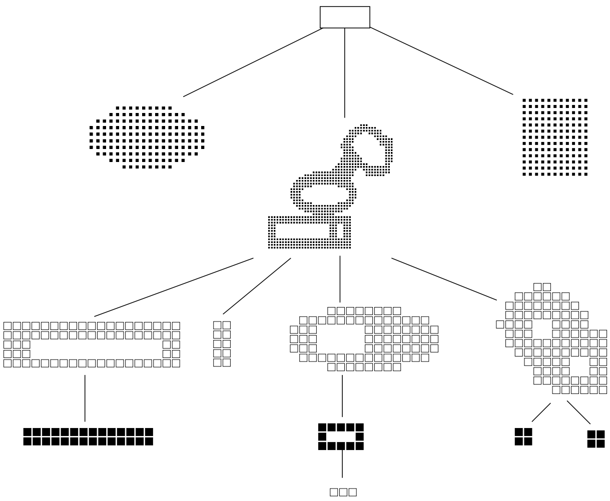
Proposition 3.2 ([91]) *Let* $\mathcal{I}$ *and* $\mathcal{I}'$ *be two digital images in* $\mathbb{Z}^2$ *whose associated adjacency trees are respectively* $\mathcal{A}(\mathcal{I})$ *and* $\mathcal{A}(\mathcal{I}')$*. Let* X *[resp.* X'*] be the set of black pixels of* $\mathcal{I}$ *[resp.* $\mathcal{I}'$*]. Then, the sets* X *and* X' *are symmetrically* $n-$*homotopic if and only if* $\mathcal{A}(\mathcal{I})$ *and* $\mathcal{A}(\mathcal{I})$ *are isomorphic.*

Note that the most difficult result which has been proved as a part of the latter Proposition is the fact that two images which have isomorphic adjacency trees can be obtained one from each other by a sequence of removal or addition of simple pixels.

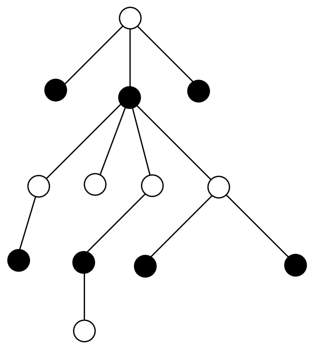
In order to avoid possible confusions between the problem of the characterization of lower homotopy and the characterization of symmetric homotopy, we give the example depicted in Figure 3.8. Here, the object of Figure 3.8(b) is obviously not lower $n-$homotopic to the object of Figure 3.8(b) since (a least) the marked pixel of Figure 3.8(a) is not $n-$simple for $n = 4$ or $n = 8$, and this remains true after any sequential removal of $n-$simple pixels in this object. This image can be found in [73] as an example of two objects which are not “topologically” equivalent. However, the words “topologically equivalent” should be understood there in the context of thinning, and in this context, topological equivalence is often used to express lower homotopy. Nevertheless, theses two objects are symmetrically homotopic in our sense as shown by Figure 3.9. In this Figure, the sets of gray pixels are simple sets (see Definition 3.2), and this show how each image can be obtained from its previous one by a sequence of deletion or addition of simple pixels. This simple example



(a) An image in $\mathbb{Z}^2$.



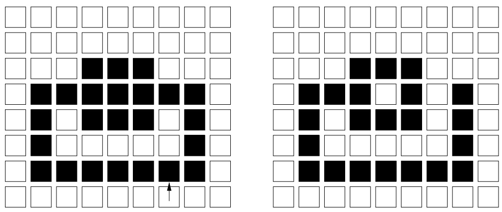
(b) The associated adjacency tree whose vertices are black and white connected components of the image. The root is the outside white connected component.



(c) A canonical representation of the adjacency tree.

Figure 3.7: Adjacency tree associated with a 2D digital image.

shows that the reader should be warned against the possible confusion between symmetric homotopy (i.e. topological equivalence) and lower homotopy.



(a) (b)

Figure 3.8: The right object is not lower $n-$homotopic to the left one for $n \in \{4, 8\}$.

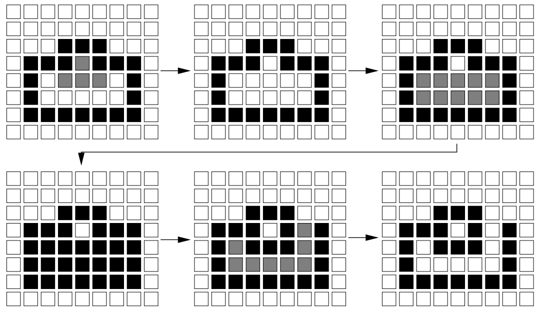


Figure 3.9: A sequence of addition and removal of $n-$simple sets (for $n \in \{4, 8\}$).

Conclusion of Part I

In this part, we have introduced the notion of digital images and provided the formalism which allows to deal with the topological properties of a digital space. Many examples have been given in the case of the digital space $\mathbb{Z}^2$; however, almost all the notions introduced have been given for any kind of digital space provided some convenient adjacency relations.

In next parts, we will investigate two other digital spaces. The first one is the space of *digital boundaries* of objects in $\mathbb{Z}^3$ and the second is $\mathbb{Z}^3$ itself. In both parts, our investigations are motivated by the use of the digital fundamental group in order to characterize topology preservation in these spaces. More precisely, our motivation is to show that the digital fundamental group is a powerful tool in this context. This goal was reached after having defined several new tools such that the *intersection number* and the *linking number*.

Part II

Topology preservation within digital boundaries

Introduction to Part II

In the 2D case, topology-preserving thinning algorithms have shown to be very basic and essential tools in pattern recognition and classification of objects represented in a planar grid. Thus, topology preservation in the 3D case is a very important question if we want to develop useful and efficient tools for 3D images analysis. Many authors have been working on homotopic thinning algorithms from which a simple definition of homotopy between digital sets can be derived (see [45], [8] or [46]). Now an open question remains about the existence of a reasonably usable algorithm to decide if a given 3D set Y is lower homotopic to another object X in which it is included. Whereas a simple necessary condition considering holes in objects of $\mathbb{Z}^2$ exists (Proposition 3.1, Part I), the 3D case is far from being so trivial. Today, a necessary condition $\mathcal{P}(X,Y)$ can be given in terms of properties of the digital fundamental group morphisms ([45]) induced by inclusion of connected component of X,Y, $\overline{X}$ and $\overline{Y}$ (see Section 2.3.2). However, one can find some examples which show that this condition is not sufficient (see "Conclusion and perspectives"). Nevertheless, in the intermediate framework of digital surfaces which will be investigated here, we will prove that the digital fundamental group is sufficient to characterize lower homotopy.

Indeed, another kind of digital objects is heavily used for image visualization and analysis : *the digital surfaces*, often called *digital boundaries* in order to avoid confusion with the surfaces defined as sets of voxels. Digital boundaries are defined as the "visible" surfaces of a 3D object when visualized as a set of unit cubes (*voxels*). These surfaces are made of unit 2D squares, the faces of the voxels, so called *surfels* (short for *surf*ace *el*ements). The data of the boundary of an object in $\mathbb{Z}^3$ is a first step for some 3D visualization, but it also provides a useful and efficient data set for image processing and analysis purposes. For example, such objects have been used in [70] by R. Malgouyres and A. Lenoir to extract some anatomical informations from Nuclear Magnetic Resonance (NMR) images.

Indeed, first works on digital surfaces were motivated by there applications in medical imagery (see [39, 41, 36]). The first motivation of the paper by Malgouyres and Lenoir was the extraction of the *loci cortical sulci* of the human brain. In order to achieve this extraction they had to apply a thinning algorithm to a binary image on the the surface of the brain; the latter image being obtained by thresholding of the image of mean curvature computed at each surfel using the method described in [53].

Indeed, the set of surfels whose curvature is negative can be seen as map from the set of surfels of the boundary to the set $\{0, 1\}$. In other words, it may be seen as an object in some kind of digital space. More generally, any property which could be checked for each surfel could lead to such an object. Then, thinning algorithms within these objects will have to consider the notions of simple surfels as intuitively defined in the previous part. Thus, this show the need of a theoretical framework for studying topology preservation within subsets of digital surfaces.

For this purpose, it is possible to define some convenient adjacency relations between surfels in such a way that a digital boundary constitutes a digital space on which all the generic notions which have been introduced in the previous part are well defined. Among other things, one can define connectivity, homotopy between paths, simple surfels and homotopy between subsets of a digital surface. In [70], the authors have proved that a similar criterion to the condition $\mathcal{P}(X, Y)$ previously mentioned, using the digital fundamental group and intersection between *cavities*, is a *necessary and sufficient* condition for lower homotopy between subsets of digital surfaces. *Cavities* here denote connected components of the complement of a part X of a digital surface Σ (Definition 1.18).

Since this latter paper, it was a conjecture that the condition about the holes was itself a consequence of the condition on the digital fundamental groups except in a very particular case. The purpose of this part is to state and prove this result and then give a new theorem about lower homotopy between subsets of a digital surface (Theorem 13 in Chapter 9). This leads to a very concise new characterization which shows the ability of the digital fundamental group to completely characterize lower homotopy in this field.

However, the lack of tools for studying homotopy classes of paths (i.e elements of the digital fundamental group) brings us to define an use a new tool : the *intersection number* which was introduced by the author and R. Malgouyres in [32]. This number, which counts the number of oriented intersections between two kinds of paths, is shown to be invariant under any homotopic deformation of the paths. Then, it can be used for example to prove that two paths are not homotopic. Especially, it can also be used to prove that a path is not reducible. This latter property can be seen as a generalization of some notions introduced in [93] where Rosenfeld and Nakamura have been studying the properties of simple closed curves in 2D, considering for example curves surrounding a 2D hole. However, in our purpose, we are interested by general digital paths, surrounding cavities but also *tunnels* which are not met in 2D digital topology. The notion of a tunnel, not simple to define, will be introduced in Chapter 5.

In Chapter 4, we will give the definition of a digital surface and introduce the basic definitions which will allow to discuss on the topological properties of such a digital space. Then, in Chapter 6 we will introduce the intersection number and state and prove its main properties. Next, two applications of this new tool will be given in Chapter 7 and 8, respectively, a new Jordan theorem within digital surfaces and a new property for the 2D winding number which has been introduced in Section 2.1.2. Finally, we will end with the main result of this part in Chapter 9 : the proof of a new theorem about the characterization of lower homotopy within digital surfaces.

Chapter 4

Digital boundaries (surfaces)

4.1 Two kinds of digital surfaces

In subsection 2.1.2, we have mentioned the $\mathcal{MA}-$surfaces which were defined by R. Malgouyres in [58]. In fact, these digital surfaces belong to the family of *thin* and *separating* subsets of $\mathbb{Z}^3$ which were defined as digital surfaces by several authors. Briefly, the idea of these definitions was to give some local characterizations which guarantee that an object, the voxels of which all satisfy these local characterizations, will have a thinness property and also separates its background in two connected components (see Definition 3.7 and Theorem 4 for the 2D analogue to the separating notion). The thinness property being enunciated as follows : each voxel of the object is adjacent to both components of its background. Such objects are usually called *strongly separating sets.* Now, R. Malgouyres proved in [60] that no local characterization of the full class of strongly separating sets can be found. Thus, there is no hope to find an analogue in $\mathbb{Z}^3$ to the local characterization of a simple closed curve given by A. Rosenfeld in [89] which is the two dimensional analogue of the class of strongly separating sets (see subsection 2.1.3).

However, several definitions have been introduced for some more restrictive object classes than the strongly separating one which, themselves, admit some local characterizations. Thus, D.G. Morgenthaler and A. Rosenfeld first introduced the notion of a *simple closed* $n-$*surface* for $n \in \{6, 26\}$ in [76]. Later, in [61], R. Malgouyres introduced the definition of the $\mathcal{MA}-$surfaces, and finally, more recent works with G. Bertrand leaded to the definition of *strong surfaces* which are proved to be the less restrictive ones (see [67, 68]).

Now, we are here interested in this part by a definition of digital surfaces which is different from the latter ones. Indeed, in this Part, we no longer consider surfaces as subsets of $\mathbb{Z}^3$ but as particular subsets of $\mathcal{R}_6$. Indeed, we will show in the following that some of these subsets can be considered as a kind of digital space with some convenient adjacency relations.

4.2 Digital surfaces, surfels

Digital surfaces are made of unit square faces, called *surfels*, which are the basic elements of the visible parts of an object of $\mathbb{Z}^3$ when depicted as a set of voxels.

Definition 4.1 (surfel) *Let $B \subset \mathbb{Z}^3$, then a couple (a, b) such that $a \in B$, $b \in \overline{B}$ and a is $6-$adjacent to b, is called a* surfel *of B.*

A surfel is in fact the common face shared by two $6-$adjacent voxels, the first one belonging to the object, the second one to its background (see Figure 4.1). Note that such a face is oriented according to the outward normal from its center and this definition of a surfel is close to the classical one which can be found for example in [103], and restricted to the 3D case. In fact, we call a *voxel face* the unit square shared by any two $6-$adjacent voxels, but a surfel is the oriented common face shared by two $6-$adjacent voxels, where the first one is a voxel of an object and the second one is a voxel of the complement of this object. A voxel may be associated to at most six surfels (see Figure 4.2) depending on the value of its $6-$neighbors, and each surfel has four *edges* and four *vertices* as depicted in Figure 4.3.

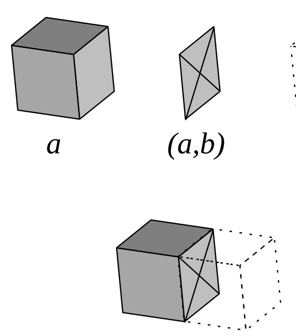


Figure 4.1: A surfel (a, b), common face shared by a black voxel and a white one.

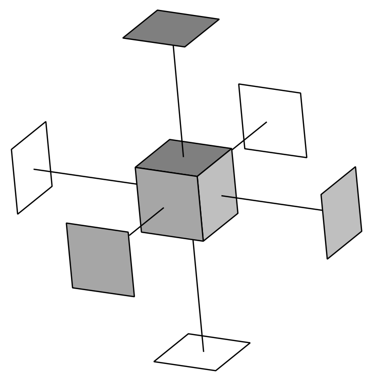


Figure 4.2: A voxel may be associated with at most six surfels.

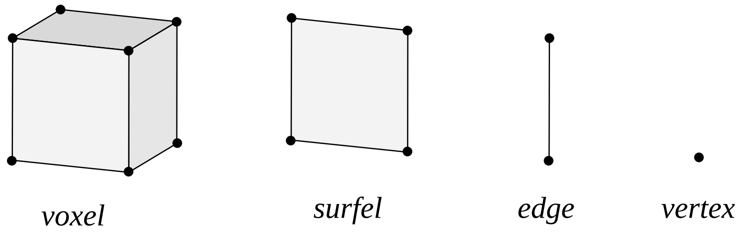


Figure 4.3: Few terms

Now, we introduce the notion of the *border* between a black and a white connected component of a digital image in $\mathbb{Z}^3$, using the same notation as in [104].

Definition 4.2 (border) *Let* $\{n, \overline{n}\} = \{6+, 18\}$. *Then, let* O *be an* $n-$*connected subset of* $\mathbb{Z}^3$ *and* V *be one of its background components. We define the* $(n, \overline{n})-$border *between* O *and* V, *denoted by* $\delta_n(O, V)$ *:*

$$\delta_n(O, V) = \{(a, b) \mid a \in O,\ b \in V \text{ and } a \text{ is } 6-\text{adjacent to } b\}$$

In the sequel of this part O and V are two subsets of $\mathbb{Z}^3$ which satisfy the conditions of Definition 4.2 and $\Sigma = \delta_n(O, V)$ is the $(n, \overline{n})-$border between O and V for $n \in \{6+, 18\}$ which will be called a *digital surface*.

The digital surface Σ has the Jordan property, according to the definition given in [103]. Indeed, any $6-$path $\pi = (y_k)_{k=0,\ldots,p}$ from a voxel y_0 of O to a voxel y_p of V is such that there exists $j \in \{0, \ldots, p-1\}$ with $y_j \in O$ and $y_{j+1} \in V$, in other words π *exists through* Σ.

Observe that this definition is very close to the definition of a cellular complex. However, an important property of a digital surface is that its surfels are constrained by some geometrical consideration in $\mathbb{Z}^3$. Reader should keep in mind that a surfel is nothing but one of the following couples for some $(x, y, z) \in \mathbb{Z}^3$:

$$((x, y, z), (x \pm 1, y, z)),\ ((x, y, z), (x, y \pm 1, z)),\ ((x, y, z), (x, y, z \pm 1)).$$

4.3 Adjacency relations between surfels

In this section, we introduce the two adjacency relations between surfels which will allow the definition of further tools used in our study of the topological properties within digital surfaces.

4.3.1 $e-$adjacency relation

A surfel in a digital surface shares a given edge with at most three other surfels as depicted in Figure 4.4 (this result is generalized to digital boundaries in $\mathbb{Z}^n$ in Proposition 3.5 of [103]). This leads to the definition of the following relation $\mathcal{R}_{edge}$:

Definition 4.3 ($\mathcal{R}_{edge}$) *We define the adjacency relation $\mathcal{R}_{edge}$ in Σ as follows :*

$$\mathcal{R}_{edge} = \left\{ ((a,b),(a',b')) \;\middle|\; \begin{array}{l} (a,b) \in \Sigma,\ (a',b') \in \Sigma,\ (a,b) \neq (a',b') \\ a, b, a' \text{ and } b' \text{ belong to a common } 2 \times 2 \text{ square of } \mathbb{Z}^3 \end{array} \right\}$$

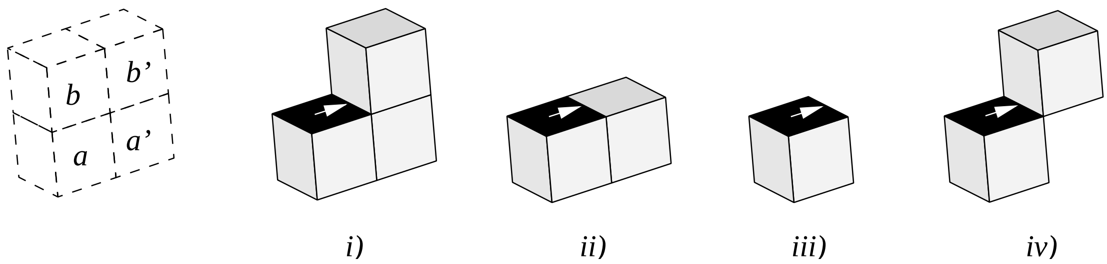


Figure 4.4: The surfel (a, b) may share one of its edges with at most 3 other surfels.

Thus, the surfel (a, b) of Figure 4.4 shares the marked edge either with the surfel (b', b) (case i), with the surfel (a', b') (case ii), with the surfel (a, a') (case iii), or simultaneously with the surfels (b', b), (b', a') and (a, a') in the case iv. This relation is an adjacency relation on the digital space Σ. However, as recalled in the introduction of this part, we are interested by the characterization of topology preservation for subsets of digital surfaces. But it has appeared that no convenient topological properties can be derived for this digital space together with the $\mathcal{R}_{edge}$ adjacency relation. Indeed, the adjacency relations used for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$ have the good properties to be strictly related to the space distribution of the spels. However, the relation $\mathcal{R}_{edge}$ leads to some unusable adjacency graphs. Furthermore, connectivity between surfels was first introduced in order to design some efficient boundary tracking algorithms which allow the retrieval of the visible boundaries of an object of $\mathbb{Z}^3$. For this purpose, $\mathcal{R}_{edge}$ also has important drawbacks. Indeed, the fact that too many paths in the adjacency graph pass through a given surfel prevents the existence of an efficient algorithm based on local computation which could retrieve the boundary of an object. However, we can define an adjacency relation $\mathcal{R}_e$ between surfels, which depends on the $n-$adjacency relation considered for the object ($n \in \{6+, 18\}$), in such a way that a surfel has exactly four $\mathcal{R}_e-$neighbors, i.e.

exactly one per edge . The definition of this classical regular graph on Σ can be found for instance in [92] and is as follows :

Definition 4.4 ($\mathcal{R}_e$) *Let (a,b) and (a',b') be two surfels of Σ. The surfels (a,b) and (a',b') are said to be $\mathcal{R}_e-$*adjacent *if $((a,b),(a',b')) \in \mathcal{R}_{edge}$ and :*

• *If $(n,\overline{n}) = (6+,18)$ then a and a' are $6-$connected by a $6-$path with a length $l < 3$ in O.*

• *If $(n,\overline{n}) = (18,6+)$ then b and b' are $6-$connected by a $6-$path with a length $l < 3$ in V.*

A pair $\{(a,b),(a',b')\}$ of $\mathcal{R}_e-$adjacent surfels of Σ is called an edgel.

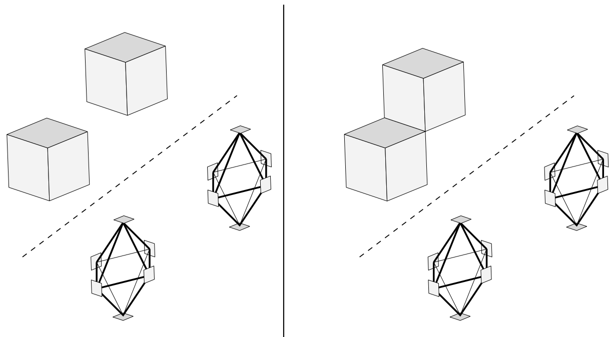
Notation 4.1 In the sequel, and in order to avoid heavy notations, we will abbreviate the adjacency relation "$\mathcal{R}_e$" by simply "e", abbreviating for example $\mathcal{R}_e-$adjacency by $e-$adjacency.

It is important to see that the definition of the $e-$adjacency relation follows the topological properties of the object *wrapped* in the surface. Indeed, the following theorem which is in fact a justification of the boundary tracking algorithms, was proved by E. Artzy in [4] and using topological consideration by G.T. Herman in [41]. This theorem will be be useful in the sequel.

Theorem 6 *Σ is $e-$connected.*

Now, another approach to the $e-$connectivity property of Σ can be given. Indeed, we defined Σ as the set of spels between an $n-$connected set of voxels and one of its ($\overline{n}-$connected) background components. Then, an important result is that Σ is $e-$connected. But a converse property is also true. Let $B \subset \mathbb{Z}^3$ and S be the set of the surfels of B. Then, any $e-$connected component of surfels of S coincides with some border $\delta_n(I,E)$, where I is an $n-$connected component of B and E is an $\overline{n}-$connected component of $\overline{B}$. Thus, the objects of Figure 4.5(a) and 4.5(b) are both not $(6+)-$connected so that the set of their surfels is not $e-$connected, following the definition of $e-$connectivity in the case when $(n,\overline{n}) = (6+,18)$. Now, the object depicted in Figure 4.6 is $18-$connected so that the set of its surfels is $e-$connected for $(n,\overline{n}) = (18,6+)$.

Furthermore, these two objects are made of two $6-$connected components with no cavity. More generally, it is clear that for any object B in $\mathbb{Z}^3$ and any couple $(n,\overline{n}) \in$



(a) (b)

Figure 4.5: Two objects of $\mathbb{Z}^3$ and the $e-$adjacency graphs of their sets of surfels for $n = 6+$.

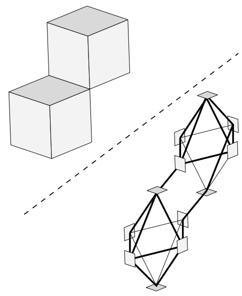
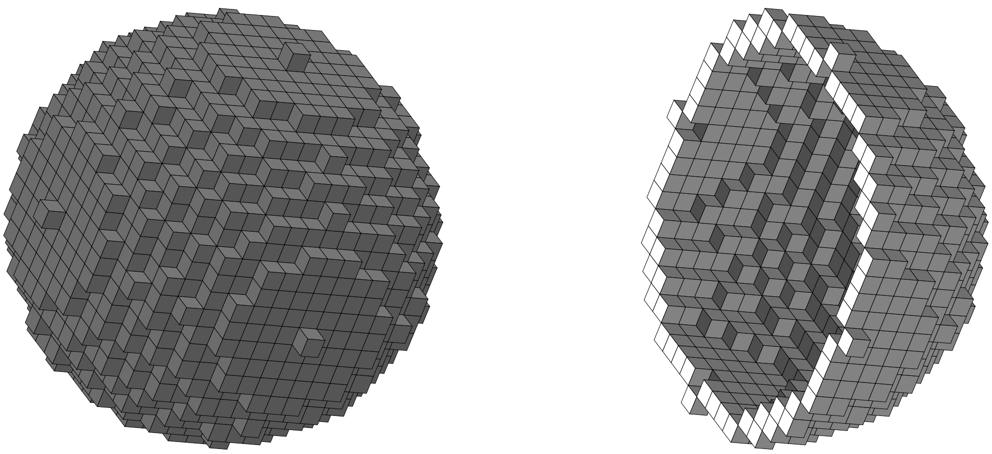


Figure 4.6: An object of $\mathbb{Z}^3$ and the $e-$adjacency graph of its set of surfels for $n = 18$.

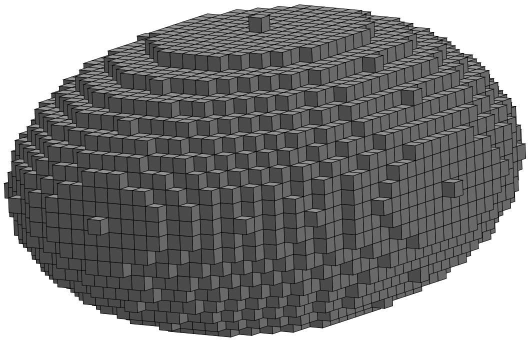
$\{(n, \overline{n}), (\overline{n}, n)\}$, the number of $e-$connected components of the set of surfels of B (i.e the number $(n, \overline{n})-$borders) is equal to the number of $n-$connected components of B plus the number of $\overline{n}-$connected components of $\overline{B}$ minus one. For example, the hollow ball depicted in Figure 4.7(a) has two $(18, 6+)-$borders whereas the object depicted in Figure 4.8 has three ones.



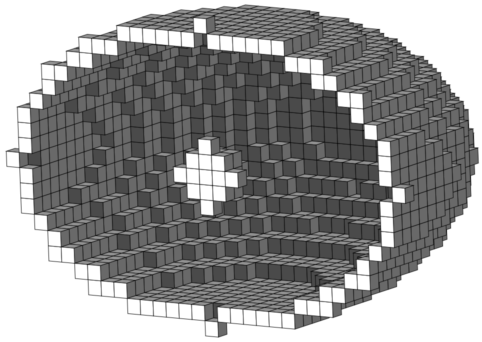
(a) View of the outside surface. (b) A cut view of the hollow ball.

Figure 4.7: A hollow ball in $\mathbb{Z}^3$ for $n = 18$.

Now, following Definition 4.4, $e-$paths and $e-$connectivity within digital surfaces are well defined following the generic definitions given in Section 1.1.



(a) View of the outside surface.



(b) A cut view of the object.

Figure 4.8: An object of $\mathbb{Z}^3$ with two connected components and a cavity for $n = 18$.

4.3.2 $v-$adjacency relation

Now, an object in a digital surface can be seen as a binary picture drawn on the boundary of a subset of $\mathbb{Z}^3$ as depicted in Figure 4.9. Then, in our investigations of the topological properties of such pictures, we should expect that some basic properties hold, like for example some kind of a Jordan property which was described for the spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$ in Chapter 2. Thus, let C be the set of black surfels of Figure 4.9. This set is a simple closed $e-$curve according to Definition 1.13 and one should expect that such a set, in this particular case[1], separates its background (the set of white surfels) in two distinct components. However, as it occurred in $\mathbb{Z}^2$, this is obviously not true if $e-$adjacency is used for the background of the image since the three surfels marked with a cross are not pairwise $e-$connected so that the background of C has in this case four $e-$connected components.

Again, this shows the need of the introduction of another adjacency relation between surfels which links two surfels which share a vertex, in other words, an analogue to the $8-$adjacency relation in $\mathbb{Z}^2$. Indeed, the $e-$adjacency in a planar digital surface coincides with the $4-$adjacency relation in $\mathbb{Z}^2$ as illustrated by Figure 4.10. However, the following definition will not be satisfying.

Definition 4.5 **($\mathcal{R}_{vertex}$)** *We define the adjacency relation $\mathcal{R}_{vertex} \subset \Sigma \times \Sigma$ as follows :*

$$\mathcal{R}_{vertex} = \left\{ ((a,b),(a',b')) \;\middle|\; \begin{array}{l} (a,b) \in \Sigma,\ (a',b') \in \Sigma,\ (a,b) \neq (a',b') \\ a, b, a' \text{ and } b' \text{ belong to a common } 2{\times}2{\times}2 \text{ cube of } \mathbb{Z}^3 \end{array} \right\}$$

In other words, the surfels (a,b) and (a',b') are $\mathcal{R}_{vertex}-$adjacent if they share a vertex.

[1]The purpose of Chapter 7 is to precise why this case can be said to be particular

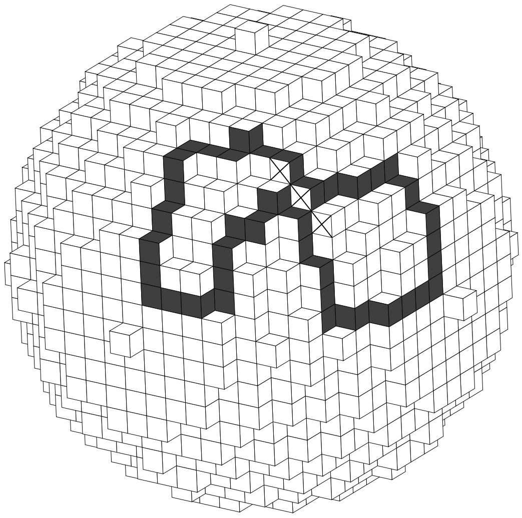


Figure 4.9: In black, a simple closed $e-$curve C of surfels, the background of which is made of four $e-$connected components.

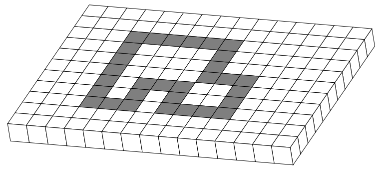


Figure 4.10: The upper part of the surface can be seen as a rectangle of $\mathbb{Z}^2$.

Remark 4.1 *The data of one vertex p of a surfel (a, b) is equivalent to the data of one of the four $2\times2\times2$ cubes which contain the two voxels a and b. Indeed, this $2\times2\times2$ cube will also contain any two voxels a' and b' such that (a', b') share with $(a, b,)$ the vertex p.*

However, this relation again has the important drawback that it links some sets of surfels the continuous analogue of which should not be connected as depicted by Figure 4.11. In Figure 4.12, we have depicted two objects in $\mathbb{Z}^3$, in which, when $(n, \overline{n}) = (6+, 18)$, the two *voxels* marked with a cross are not $(6+)-$adjacent. However, in the case of Figure 4.12(a), the two *surfels* marked with a cross are $\mathcal{R}_{vertex}-$adjacent whereas they are not in Figure 4.12(b). Since this is obviously not satisfying, we define a more restrictive adjacency relation $\mathcal{R}_v$.

Definition 4.6 ($\mathcal{R}_v$) *Let (a, b) and (a', b') be two surfels of Σ. Then $((a, b), (a', b')) \in \mathcal{R}_v$ if and only if $((a, b), (a', b')) \in \mathcal{R}_{vertex}$, and there exists an $e-$path $((a_0, b_0), \ldots, (a_k, b_k))$ in Σ such that $(a_0, b_0) = (a, b)$, $(a_k, b_k) = (a', b')$ and for $i = 1, \ldots, k-1$, $((a_k, b_k), (a, b)) \in \mathcal{R}_{vertex}$ and $((a_k, b_k), (a', b')) \in \mathcal{R}_{vertex}$.*
It is then immediate that the surfels in $\{(a_0, b_0), \ldots, (a_k, b_k)\}$ all share a unique vertex since each surfel (a_i, b_i) for $i \in \{0, \ldots, k\}$ belongs to the only 2×2×2 cube which contains the four voxels a, b, a' and b'.

Now, we give another but equivalent definition of the $\mathcal{R}_v-$adjacency relation which uses

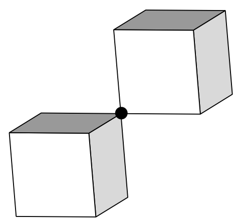
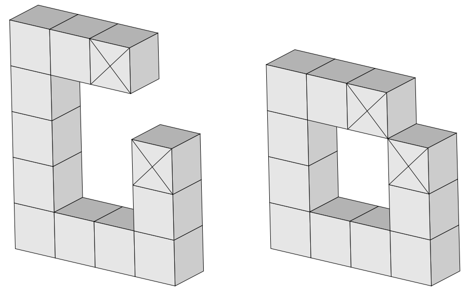


Figure 4.11: Two voxels which are neither (6+)−adjacent nor 18−adjacent whereas they define some surfels which are $\mathcal{R}_{vertex}$−adjacent.



(a) (b)

Figure 4.12: Two objects the digital surfaces of which have the same topological properties for $(n, \overline{n}) = (6+, 18)$.

the definition of a *loop* in a digital surface. Furthermore, the notion of a *loop* will be useful in the sequel since the set of surfels it defines will be used as the elementary deformation cell (see the definition of such cells in section 2.2.2) within digital surfaces.

Definition 4.7 (Loop) *A* loop $\mathcal{L}$ *in* Σ *is an* $e-$*connected set of surfels which share a given vertex* p. *Now, let* x *be a surfel of* Σ, a *be an edge of* x *and* p *be a vertex of* a. *Then, there exists a single surfel* x', $e-$*adjacent to* x *and which share the vertex* p *and the edge* a *with* x. *Now, a third surfel* $x'' \neq x$, $e-$*adjacent to* x' *shares* p *with* x *and* x'. *By repeating this process, it is possible to build a simple closed* e$-$*path* π *from* x *to* x *such that* $\pi^* = \mathcal{L}$ *is a loop of* Σ. *The path* π *is called a* parameterization of the loop $\mathcal{L}$.

Now, the following definition is equivalent to Definition 4.8.

Definition 4.8 ($\mathcal{R}_v$) *Let* $x \in \Sigma$ *and* $y \in \Sigma$. *The two surfels* x *and* y *are said to be* $\mathcal{R}_v-$*adjacent if there exists a loop* $\mathcal{L}$ *of* Σ *such that* $\{x, y\} \subset \mathcal{L}$.

Then, from the very definitions of the $e-$adjacency and the loops, we obtain that the two surfels marked with a cross in Figure 4.12(b) are $\mathcal{R}_v-$adjacent if Σ is an $(18, 6+)-$border whereas they are not $\mathcal{R}_v-$adjacent if Σ is a $(6+, 18)-$border.

Notation 4.2 In the sequel, and in order to avoid heavy notations, we will abbreviate the relation "$\mathcal{R}_v$" by simply "v".

One can see that a vertex is not sufficient to uniquely define a loop since a vertex can be associated with two distinct loops. Indeed, if we consider the $(18, 6+)-$border Σ of the object of Figure 4.13, and where all the voxels are visible; then the black vertex defines two loops. The first one is made of the six visible surfels which share this vertex and the second one is made of three hidden surfels. By the same way, it we consider the $(6+, 18)-$border of the same object, this vertex defines three distinct loops of three surfels each.

4.4 The fundamental group in a digital surface

At this step, all the notions which have been introduced in Part I for a digital space together with some complementary adjacency relations ($e-$adjacency and $v-$adjacency) are well defined except homotopy of paths (and so neither the digital fundamental group). Following the steps of the definition given in subsection 2.2.2, we must first define elementary deformation cells for digital surfaces.

Notation 4.3 In the sequel of this Part, and when no more precision is given, the prefix "$n-$" stands for either "$e-$" or "$v-$", and no more for any adjacency relation in $\mathbb{Z}^2$ or $\mathbb{Z}^3$. We may shorten this by writing $n \in \{e, v\}$.

4.4.1 Deformation cell and assumption about Σ

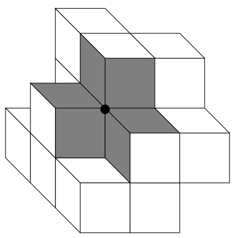


Figure 4.13: Example of a loop.

Definition 4.9 (elementary deformation cell in Σ) *An* elementary $n-$deformation cells *(for $n \in \{e, v\}$) in a digital surface Σ is a loop of Σ.*

Now, the latter definition leads to the following formulation of Definition 2.10 : two $n-$paths in a subset X of a digital surface Σ with same extremities are said to be equivalent up to an elementary $n-$deformation if they are the same but in a loop of Σ (see Figure 4.14).

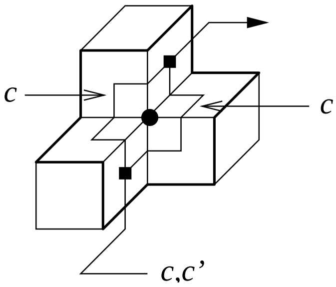


Figure 4.14: Illustration of an elementary $e-$deformation.

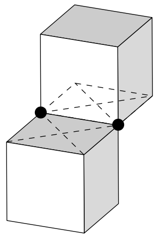
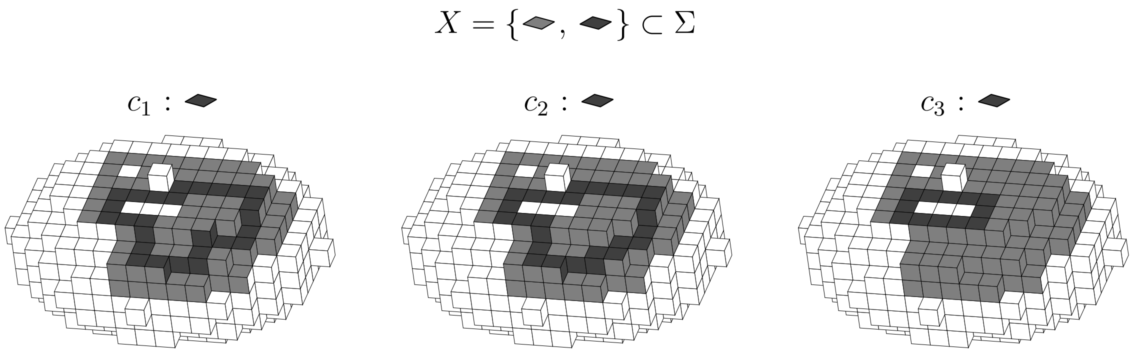


Figure 4.15: The two surfels of this $(18,6+)-$border marked with a cross are not $e-$adjacent but belong to two distinct loops.

Definition 4.9 also allows to define $\mathcal{R}-$homotopy for paths of surfels where $\mathcal{R} \in \{\mathcal{R}_e, \mathcal{R}_v\}$ as illustrated in Figure 4.16. Thus, the $e-$path c_1 of Figure 4.16(a) is an elementary $e-$deformation of the $e-$path c_2 of Figure 4.16(b). Finally, the path c_1 is $e-$homotopic to the $e-$path c_3 of Figure 4.16(c).



(a) (b) (c)

Figure 4.16: Example of $e-$homotopic paths in a subset X of a digital surface Σ.

Now, in the case when $\Sigma = \delta_{(18,6+)}(O,V)$ (i.e. O is $18-$connected), we avoid some special configurations by the assumption that any loop of the surface is a topological disk so that the situation depicted in Figure 4.17 may not occur (see [70]). A formal way to express this assumption is to say that two $v-$adjacent surfels which are not $e-$adjacent cannot both belong to two distinct loops (see Figure 4.15). An equivalent formulation can be stated as follows : we assume that $\Sigma = \delta_{18}(O,V)$ and if there exists in O two $18-$adjacent voxels which are not $6-$adjacent then, at least one of the two following properties is satisfied :

- The two voxels have an 18−neighbor in O in common.
- The voxels have two common 26−neighbors in O which are themselves 26−adjacent.

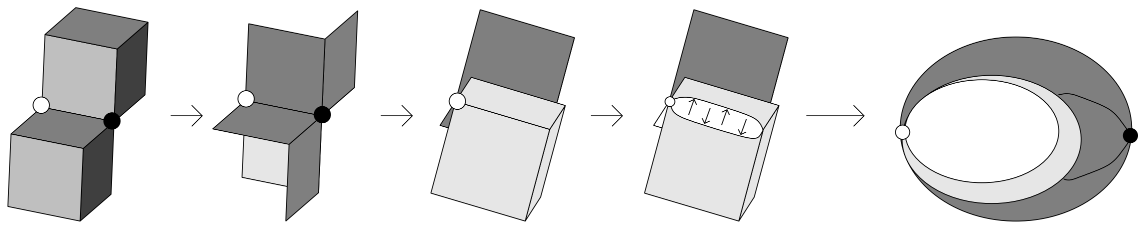


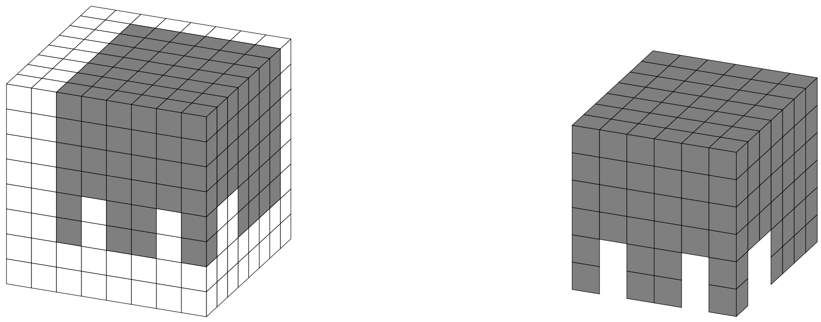
Figure 4.17: A case when the continuous analogue of a loop is not a topological disk

This restriction is necessary and sufficient to ensure that a loop is a topological disk. We need a similar restriction on $\overline{O}$ when $\Sigma = \delta_{6+}(O, V)$. In the sequel, we refer to the following remark :

Remark 4.2 *Exactly one loop of Σ may contain two surfels which are $v-$adjacent but not $e-$adjacent.*

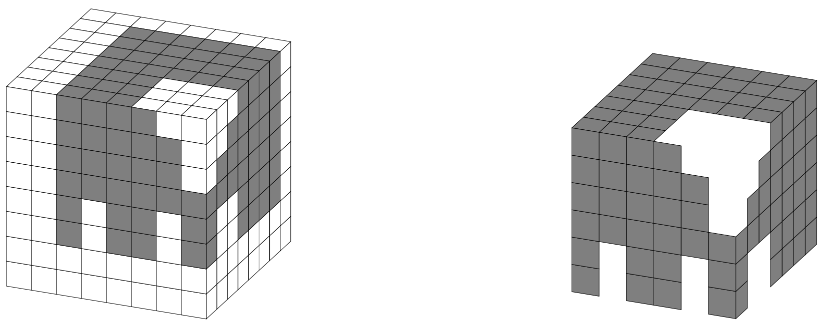
4.4.2 Some examples

In the sequel of this part we will prove that the digital fundamental group, which can be used to define simple surfels in a digital surface, is also sufficient to characterize lower homotopy in this space. In order to illustrate this notion, we give here some examples of objects in a digital surface whose fundamental groups are not isomorphic. Thus, the object X_1 depicted in Figure 4.18 is simply connected according to Definition 2.13. In other words, for any surfel $x \in X_1$, the digital fundamental group $\Pi_1^n(X_1, x)$ is reduced to the class of the trivial path (x, x). Then, the object X_2 of Figure 4.19 is not simply connected since it is clear for any surfel $x \in X_2$, there exists a closed path from x which is not reducible in X_2. In the object X_3 of Figure 4.20, and for any surfel $x \in X_3$ one can show that there exists three elements in $\Pi_1^n(X_3, x)$ such that each one cannot be expressed as a product involving the two others.



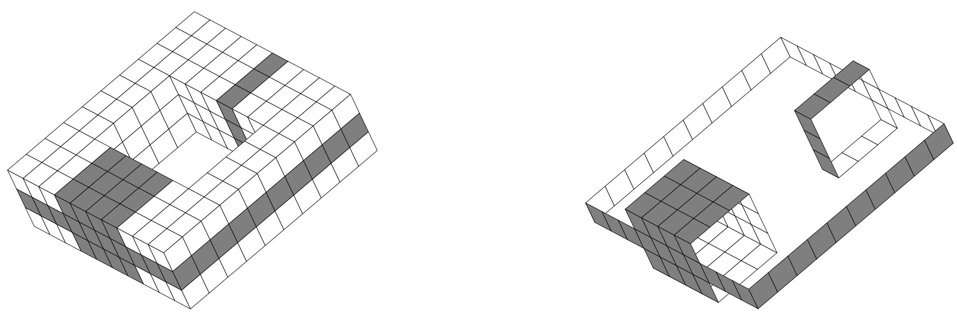
(a) In grey, a subset $X_1 \subset \Sigma_1$. (b) The set X_1.

Figure 4.18: X_1 is a simply $n-$connected set of surfels.



(a) In grey, a subset $X_2 \subset \Sigma_2$. (b) The set X_2.

Figure 4.19: X_2 is not simply $n-$connected.



(a) In grey, a subset $X_3 \subset \Sigma_3$. (b) The set X_3.

Figure 4.20: X_2 is not simply $n-$connected. Furthermore, for any $p_2 \in X_2$ and $p_3 \in X_3$, the groups $\Pi_1^n(X_2, p_2)$ and $\Pi_1^n(X_3, p_3)$ are not isomophic.

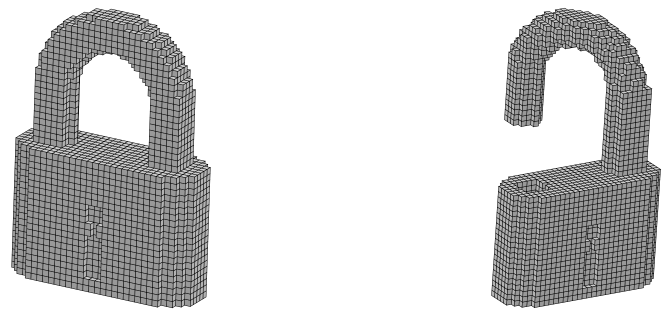
Chapter 5

Topology preservation within digital surfaces

In this chapter we will recall some works of R. Malgouyres and A. Lenoir about the definition and the characterization of topology preservation for subsets of a digital surface. Observe that such works were motivated by the formal justification of the thinning algorithm they used in order to extract some anatomical data from an image of the mean curvature computed at the surface of the brain.

In [70], Malgouyres and Lenoir gave three equivalent characterizations of homotopy, which was defined using a classical definition of simple surfels using local connectivity properties. In the context of digital surfaces, as well as in the context of 3D objects, homotopy can be understood as a particular case of *deformation retract* as defined for cellular complexes. Now, in the field of digital surfaces, topology preservation and especially lower homotopy becomes harder to characterize compared to the 2D case, this because digital surfaces and their subsets may have *tunnels*. Indeed, whereas subsets of $\mathbb{Z}^2$ were fully topologically characterized using some connectivity considerations only, subsets of digital surfaces, as well as subsets of $\mathbb{Z}^3$, are also characterized by the number and the position of their tunnels. Tunnel are not simple to define in a formal way, nevertheless some intuitive idea of what they are can be given. Indeed, we will say that the object of Figure 5.1(a) has a tunnel whereas the object of Figure 5.1(b) does not. Furthermore, it is clear that such a property cannot be characterized by the only use of connectivity considerations. Indeed, each of previous objects is connected and has no cavity.

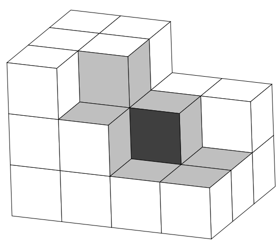
In this chapter, X is a subset of a digital surface Σ.



(a) (b)

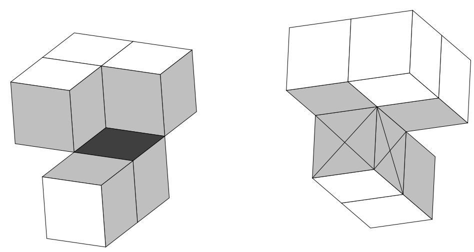
Figure 5.1: Illustration of the notion of a tunnel.

5.1 Simple surfels and homotopy



: x, : $N_v(x)$

Figure 5.2: The $v-$neighborhood of a surfel.



(a) Front view (b) Back view

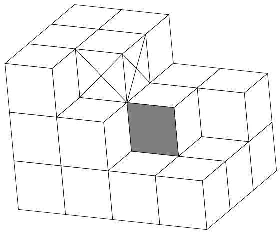
: x, : $N_v(x)$

Figure 5.3: A surfel the $v-$neighborhood of which is not a topological disk.

Let x be a surfel of Σ. As previously set in Section 4.4, we assume that any loop in Σ is a topological disk. However, the $v-$neighborhood of the surfel x is not always a topological disk. In Figure 5.2 we have depicted the $v-$neighborhood $N_v(x)$ of a given surfel and in Figure 5.3 we have depicted a surfel the neighborhood of which is not a topological disk since the two surfels marked with a cross in this Figure 5.3(b) are $e-$adjacent. In such a case, we have to define a "topology" on $N_v(x) \cup \{x\}$ under which it is a topological disk. Then, we will use a particular graph in $N_v(x)$ defined using the following adjacency relation.

Definition 5.1 (n_x-adjacency relation) *Let $x \in \Sigma$ and let y and y' be two surfels of $N_v(x) \cup \{x\}$. We say that y and y' are* n_x-adjacent *if they are $n-$adjacent and are* *contained in a common loop which contains x (see Figure 5.4).*

For example, in Figure 5.3(b), the two surfels marked with a cross are neither e_x-adjacent nor v_x-adjacent if x is the black surfel of Figure 5.3(a).

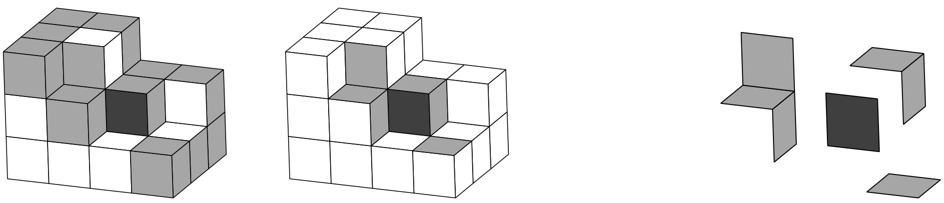


$: x$

Figure 5.4: The two surfels marked with a cross are e_x-adjacent.

Definition 5.2 ($G_n(x, X)$) *Let $x \in \Sigma$ and $(n, \overline{n}) \in \{(e, v), (v, e)\}$. We denote by $G_n(x, X)$ the set of the n_x-connected components of $N_v(x) \cap X$. Observe that $G_n(x, X)$ is a set of subsets and not a set of surfels. We will denote by $\mathcal{C}_n^x[G_n(x, X)]$ the set of all the n_x-connected components of $G_n(x, X)$ which contain a surfel $n-$adjacent to x.*

In Figure 5.5(c) we have depicted the three e_x-connected components of $G_e(x, X)$ associated with the surfel x and the set X of Figure 5.5(a). Observe that the connected component of $G_e(x, X)$ which is reduced to a single surfel is not $e-$adjacent to x so that $\mathcal{C}_e^x[G_e(x, X)]$ has only two elements.



(a) x and X (b) $N_v(x) \cap X$ (c) x and the three e_x-connected components of $G_e(x, X)$

$x :$ $X :$

Figure 5.5: Example of a set $G_e(x, X)$.

R. Malgouyres and A. Lenoir proposed the following definition of a simpleness property for surfels (a surfel x in X is said to be $n-$interior to X if $G_{\overline{n}}(x, \overline{X}) = 0$, it is called $n-$isolated if $G_n(x, X) = 0$).

Definition 5.3 (simple surfel [70]) *A surfel x is called $n-$*simple in *X if and only if the number $Card(\mathcal{C}_n^x[G_n(x,X)])$ of n_x-connected components of $G_n(x,X)$ which are $n-$adjacent to x is equal to 1, and if x is not $n-$interior to X.*

Remark 5.1 *Similarly with the 2D case, if the surfel x is neither $n-$isolated nor $n-$interior then we have $Card(\mathcal{C}_n^x[G_n(x,X)]) = Card(\mathcal{C}_{\overline{n}}^x[G_{\overline{n}}(x,\overline{X})])$.*

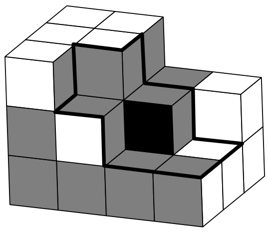
In Figure 5.6 we have depicted several examples of $v-$neighborhoods of $n-$simple and not $n-$simple surfels in a surface Σ. These examples may summarize some of the results of [70]. Indeed, we observe that the removal of the black surfel x of Figure 5.6(b) leads to the creation of an $e-$connected component in $\overline{X} \cup \{x\}$ so that this surfel is not $v-$simple for the set X considered. We also observe that the removal of the surfel x of Figure 5.6(d) would create an $e-$connected component of X for $(n,\overline{n}) = (e,v)$. Still in Figure 5.6(d) but for $(n,\overline{n}) = (v,e)$ we can say that the removing the surfel x would either disconnect the object X or remove a *tunnel* of X depending on the existence of a $v-$path in X between the surfels of $N_v(x) \cap X$ which are not $v-$connected in $N_v(x) \cap X$.

The Definition 5.3 leads to the definitions of lower and symmetric homotopy introduced in Chapter 3. Lower homotopy has been also defined in a similar way in [70] as follows :

Definition 5.4 (lower homotopy) *Let $Y \subset X \subset \Sigma$. The set Y is said to be* lower *$n-$*homotopic to *X if and only if Y can be obtained from X by* sequential *deletion of $n-$simple surfels.*

This notion of homotopy allows to define topology-preserving thinning algorithms within subsets of a digital surface. Note that this definition of *topology preservation* relies on the definition of simple surfels. The definition of a simple surfel which was previously recalled follows the classical conditions which prevent local disconnection or local re-connection. In [70], a justification of the latter definition for simple surfels is given using the Euler characteristic, saying that a surfel is simple if its contribution to the Euler characteristic of the object is equal to zero.

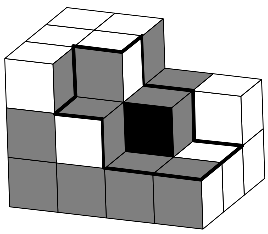
Now, the two following sections will provide some characterizations of lower homotopy for objects in a digital surface.



$e-$simple

not $v-$simple since $Card(\mathcal{C}_e^x[G_e(x, \overline{X})]) = 0$

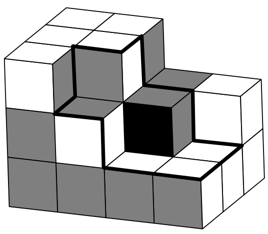
(a)



not $e-$simple since $Card(\mathcal{C}_e^x[G_e(x, X)]) = 2$

not $v-$simple since $Card(\mathcal{C}_e^x[G_e(x, \overline{X})]) = 0$

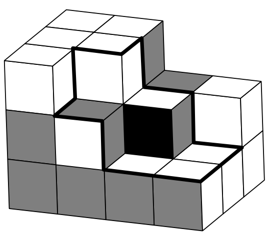
(b)



$e-$simple

not $v-$simple since $Card(\mathcal{C}_v^x[G_v(x, X)]) = 2$

(c)



not $e-$simple since $Card(\mathcal{C}_e^x[G_e(x, X)]) = 2$

not $v-$simple since $Card(\mathcal{C}_v^x[G_v(x, X)]) = 2$

(d)

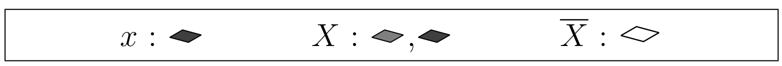
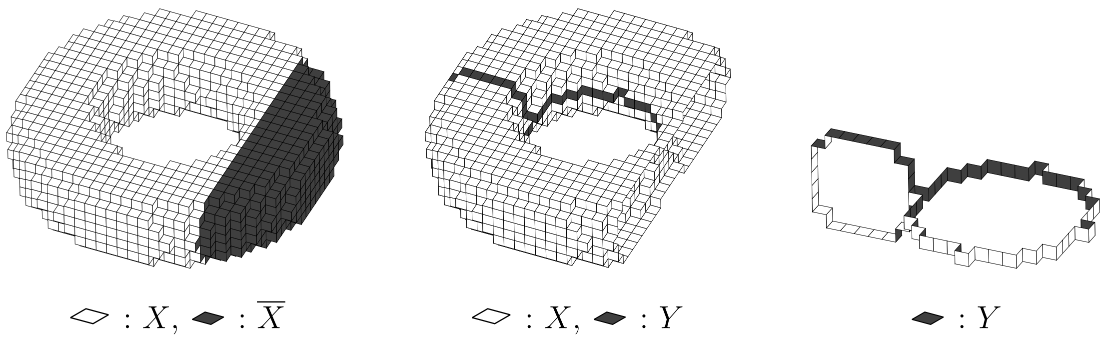


Figure 5.6: Examples of $n-$simple and not $n-$simple surfels for $(n, \overline{n}) \in \{(e, v), (v, e)\}$.



(a) (b) (c)

Figure 5.7: A subset Y of X which is lower $n-$homotopic to X for $n \in \{e, v\}$.

5.2 Euler characteristic and lower homotopy

In Chapter 9, we will need to define precisely what we call a *topological disk* and a *topological sphere* in the context of digital surfaces. For this purpose, we will use the classical notion of Euler characteristic which has been defined for this framework in [70]. Furthermore, we will recall the characterization of lower homotopy within digital surfaces which involves this classical topological invariant. In this section X is a subset of Σ and $n \in \{e, v\}$.

Definition 5.5 ($\{0,1,2\}-$cells) *We associate a dimension to surfels, edgels and loops which is equal respectively to* 2, 1 *and* 0. *We can identify a surfel x with the set $\{x\}$. We call a surfel a* $2-$cell, *an edgel a* $1-$cell *and a loop a* $0-$cell.

Definition 5.6 (Elementary Euler $n-$characteristic of a cell) *For $d \in \{0, 1, 2\}$ and for c a $d-$cell, we define the* elementary Euler characteristic *of c in X denoted by $\chi_n^d(X, c)$ as follows :*

$$\chi_n^d(X, c) = (-1)^d.Card(\mathcal{C}_n(c \cap X)).$$

Note that the only case in which $\chi_n^d(X, c)$ can be different from 0, 1 *and* -1 *is when c is a loop and $n = e$. If $\mathcal{E}_\Sigma$ and $\mathcal{L}_\Sigma$ are respectively the sets of edgels and loops of Σ, we denote :*

$$\chi_n^2(X) = \sum_{s \in \Sigma} \chi_n^2(X, s),\ \chi_n^1(X) = \sum_{\epsilon \in \mathcal{E}_\Sigma} \chi_n^1(X, \epsilon) \textit{ and } \chi_n^0(X) = \sum_{l \in \mathcal{L}_\Sigma} \chi_n^0(X, l).$$

Definition 5.7 (Euler $n-$characteristic) *We define the* Euler $n-$characteristic *of X, and we denote by $\chi_n(X)$ the following quantity :*

$$\chi_n(X) = \chi_n^0(X) + \chi_n^1(X) + \chi_n^2(X) = Card(X) + \chi_n^1(X) + \chi_n^2(X).$$

For example, the Euler $n-$characteristic associated to a set X constituted by two surfels which are $v-$adjacent but not $e-$adjacent is equal to 2 if $n = e$ $(\chi_e(X) = 2 - 8 + 8)$ whereas it will be equal to 1 if $n = v$ $(\chi_v(X) = 2 - 8 + 7)$.

The following theorem has been proved in [70], which provides a computable characterization of lower homotopy within digital surfaces :

Theorem 7 ([70]) *If $Y \subset X \subset \Sigma$ are $n-$connected, then the following properties are equivalent :*

i) Y is lower $n-$homotopic to X.

ii) $\chi_n(X) = \chi_n(Y)$ and each $\overline{n}-$connected component of $\overline{Y}$ contains a surfel of $\overline{X}$.

Thus, let X and Y be the subsets of a surface Σ as depicted in Figure 5.8. The set Y is not lower $n-$homotopic to X since, although the only $\overline{n}-$connected component of $\overline{Y}$ does contain a surfel of $\overline{X}$, its Euler $n-$characteristic (equal to zero) is different from X's one.

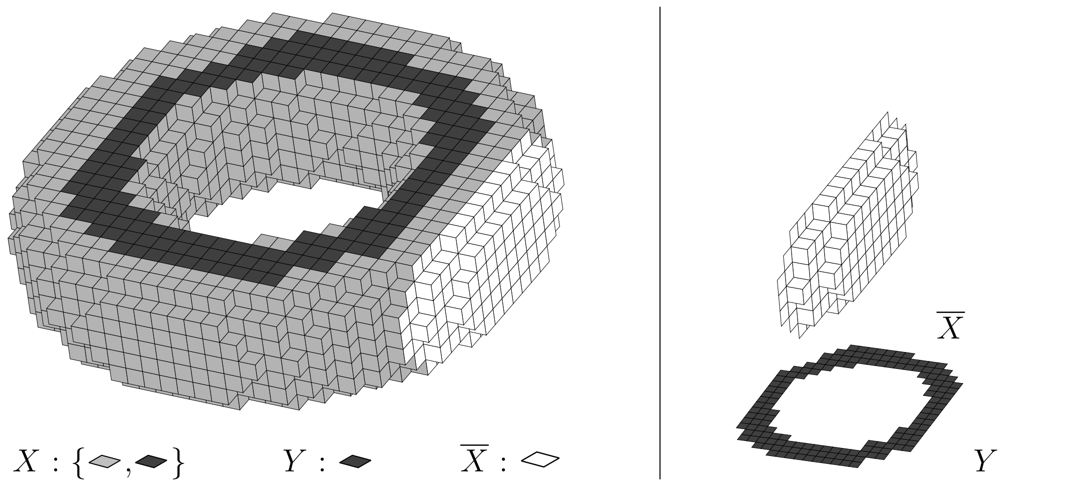


Figure 5.8: $\chi_n(X) = -1$ and $\chi_n(Y) = 0$ for $n \in \{e, v\}$.

5.3 Digital fundamental group and homotopy

Theorem 7 provides a characterization which allows us to check the lower homotopy property between two subsets of a digital surface. Now, the definition of homotopy was initially stated using the notion of simple surfel, itself derived from the definition of simple pixels in $\mathbb{Z}^2$. The characterization of lower-homotopy using the Euler characteristic then *validates* this definition and shows that it is convenient : indeed, sequential deletion of simple surfels does not change the number of connected components as well as the number of tunnels in the surface. However, the 3D case shows that the Euler characteristic, even if it has the good property of being easily computable and allows to count the number of tunnels in the surface, does not provides some information about the localization of the tunnels. This important drawback motivates the study of another characterization of topology preservation using some other topological invariants.

One purpose of [70] was to provide another characterization of lower homotopy within digital surfaces which involves the digital fundamental group. Finally, the following theorem has been proved in [70].

Theorem 8 ([70]) *Let $Y \subset X$ be two $n-$connected subsets of Σ. The set Y is lower $n-$homotopic to X if and only if the two following properties are satisfied for any surfel $B \in Y$:*

i) The morphism $i_* : \Pi_1^n(Y, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i : Y \longrightarrow X$ is an isomorphism.

ii) Each $\overline{n}-$connected component of $\overline{Y}$ contains a surfel of $\overline{X}$.

And the proof of the latter theorem uses the following Lemma.

Lemma 5.1 ([70]) *Let $X \subset \Sigma$, and let $x \in X$ be an $n-$simple surfel of X. Then, for any $B \in X \setminus \{x\}$, the group morphism $i_* :\ \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion of $X \setminus \{x\}$ in X is a group isomorphism.*

We recall the following lemma which is a straightforward consequence of Theorem 7 (see Section 5.2) and Theorem 8.

Lemma 5.2 *Let Z be an $n-$connected subset of Σ, then the following conditions are equivalent :*

i) There exists z in Z such that $\{z\}$ is lower $n-$homotopic to Z.

ii) $\overline{Z}$ has exactly one $\overline{n}-$connected component and $\chi_n(Z) = 1$.

iii) $Z \neq \Sigma$ and $\Pi_1^n(Z, B) = \{[(B, B)]\}$ for all $B \in Z$.

iv) $Z \neq \Sigma$ *and* $\chi_n(Z) = 1$.

Lemma 5.2 leads to the definitions of a *topological disk* and a *topological sphere*.

Definition 5.8 *An* $n-$*connected subset* Z *of* Σ *is called a* topological disk *if it satisfies the four conditions of Lemma 5.2.*

Definition 5.9 *If* $Z = \Sigma$ *and* $\chi_n(Z) = 2$*, we say that* Z *is a* topological sphere.

Let X and Y be the two subsets of the surface depicted in Figure 5.8. One can see that Theorem 8 allows to check that the object Y is not lower $n-$homotopic to X. Indeed, let B be the surfel marked with a cross in Figure 5.9, then the class of the path c of Figure 5.9 in $\Pi_1^n(X, B)$ can obviously not be reached by the morphism i_* induced by the inclusion of Y in X. This because no closed path of $A_n^B(Y)$ is $n-$homotopic to the closed path c in X. Then, i_* is not onto and, Theorem 8 implies that the set Y is not lower $n-$homotopic to X.

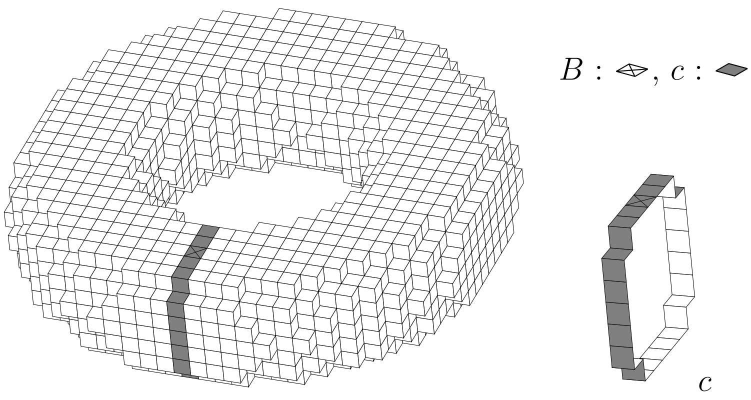


Figure 5.9: A closed path c in the set X of Figure 5.8.

5.4 Conclusion of Chapter 4 and Chapter 5

We have defined in the two previous chapters a new framework for topology preservation problems which was motivated by a practical application : homotopic thinning of a binary picture drawn on the surface of a 3D object. However, since [70], it was still a conjecture that the digital fundamental group was sufficient to characterize lower homotopy within digital surfaces. In other words, it was intuitively stated that the condition *ii*) of Theorem 8 was in fact implied by the condition *i*), except in a very particular case.

The proof of what becomes a theorem is given in Chapter 9 and was first published by the author and R. Malgouyres in [31] in a slightly different way than the one given here. Finally, the more concise version of the proof which is presented here has been submitted for publication ([30]). This proof involves a new tool the definition of which is given in the next Chapter : the *intersection number*.

In order to show the usefulness of this tool before its definition, we give here in intuitive words a way to prove the following affirmation : *"This because no closed path of $A_n^B(Y)$ is $n-$homotopic to the closed path c."*. Indeed, suppose that we have defined the number of real intersections $\mathcal{I}_{\pi,c}$ between two paths of surfels π and c (in a similar way to the definition of the winding number in Section 2.1.2). Now, suppose that we have also proved that this number cannot change when an homotopic deformation is applied to one of the two paths. Then it becomes easy to prove that no $e-$path in the set Y of Figure 5.8 is $n-$homotopic to the path c of Figure 5.9. Indeed, let π be the closed $v-$path in $\overline{X}$ depicted in Figure 5.10. It appears that $\mathcal{I}_{\pi,c} = \pm 1 \neq 0$. Now, suppose that there exists in Y an $e-$path c' which is homotopic to c in X. Then, we should have $\mathcal{I}_{c,\pi} = \mathcal{I}_{c',\pi}$. However, it is obvious that c', included in Y, has an intersection number with π equal to zero. Finally, this implies that such a path c' cannot exist.

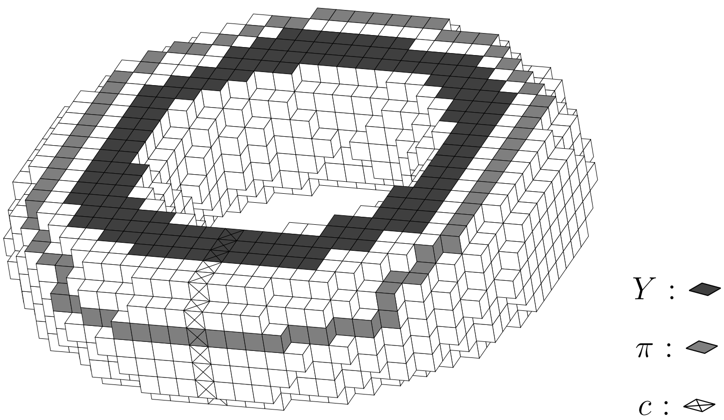


Figure 5.10: No closed path in Y can be $n-$homotopic in Σ to the closed path c.

Chapter 6

Intersection number

In this chapter, we introduce a new tool for proving theorems in the framework of digital surfaces which has been first introduced in Fourey & Malgouyres 99[32]. The main idea of this tool is to count the number of *real intersections* between a $v-$path and an $e-$path. Again, we have to use two complementary adjacencies notably in order to avoid the classical topological paradox of paths which could cross without intersecting each other. Furthermore, the intersection number will be used in proofs, for example in order to show that a given $n-$path π is not reducible in an object, and this will be possible by showing the existence of a closed $\overline{n}-$path in the object which has an intersection number with π different from zero.

Indeed, like mentioned at the end of the previous Chapter, the main property of this number is that it is left unchanged when one apply an homotopic deformation to any of the two paths. Because of this property, it may be called a new topological invariant of the digital framework.

In this chapter, Σ denotes a digital surface as defined in Chapter 4.

6.1 Definition

In order to define *oriented intersections*, we first define an orientation for surfels and then what we call the local "left side" and the local "right side" of an $n-$path.

Notation 6.1 (Vertices and oriented edges) Since a surfel has four vertices, we can order these vertices as in [92] by distinguishing one vertex for each of the six types of surfels and impose a turning order for vertices around the outward normal to the surfel (the counterclockwise order). Each vertex of a given surfel is thus associated with a number in $\{0,1,2,3\}$ (see Figure 6.1). With this parameterization of vertices we can define *oriented edges* as couples of consecutive vertices according to the cyclic order. So, for each surfel, we have the four following oriented edges : $(0,1)$, $(1,2)$, $(2,3)$ and $(3,0)$. For an $e-$path $\pi=(y_k)_{k=0,\ldots,p}$ and for $k\in\{0,\ldots,p\}$, we define $front_\pi(k)$ when $y_k\neq y_{k+1}$ [resp. $back_\pi(k)$ when $y_k\neq y_{k-1}$] as the oriented edge of the surfel y_k shared as an edge by y_k and y_{k+1} [resp. y_k and y_{k-1}]. Note that $back_\pi(0)$ and $front_\pi(p)$ are not defined if π is not closed.

Notation 6.2 For a surfel x and a given vertex number $w\in\{0,1,2,3\}$ we denote by $\mathcal{L}_w(x)$ the unique loop associated with the vertex w of x which contains the surfel x.

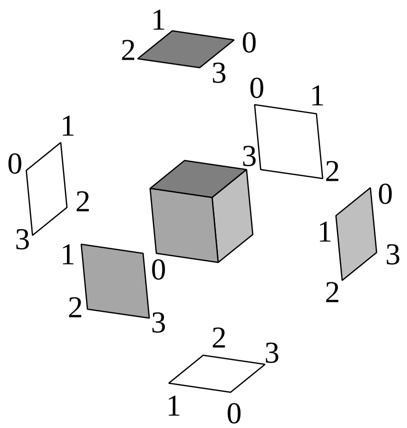


Figure 6.1: Parameterization of the vertices for each type of surfel so that the oriented edges are $(0,1)$, $(1,2)$, $(2,3)$ and $(3,0)$.

The next lemma will allow us to introduce the notion of a canonical parameterization of the $v-$neighborhood of a surfel.

Lemma 6.1 *Let x be a surfel of Σ. Then, $N_v(x)$ is a simple closed e_x-curve.*

Proof : We prove that for any surfel y of $N_v(x)$, there are exactly two surfels z_1 and z_2 in $N_v(x)$ such that y is e_x-adjacent to z_1 and z_2.

Let y be a surfel of $N_v(x)$. First, we suppose that y and x are not $e-$adjacent. Then, from Remark 4.2, only one loop $\mathcal{L}$ of Σ may contain both x and y. Let z_1 and z_2 be the only (from the very definition of a loop) two surfels of $\mathcal{L}$ which are $e-$adjacent to y. It follows that z_1 and z_2 are the only two surfels $e-$adjacent to y which can belong to a loop which contains x. In other words, z_1 and z_2 are the only two surfels e_x-adjacent to y.

Now, we may suppose that x and y are $e-$adjacent. Then, we may suppose without loss of generality that y shares as an edge with x the oriented edge $(0,1)$ of y. Then, exactly two loops contain both x and y : $\mathcal{L}_0(y)$ and $\mathcal{L}_1(y)$. Let z_1 be the (unique) surfel of Σ which shares as and edge with y the oriented edge $(3,0)$ of y; and let z_2 be the (unique) surfel of Σ which shares as and edge with y the oriented edge $(1,2)$ of y. It is immediate that $\{z_1, z_2\} \subset N_v(x)$ and $z_1 \neq z_2$. Obviously the surfel z_3 which share as an edge with y the oriented edge $(2,3)$ can belong neither to $\mathcal{L}_1$ nor to $\mathcal{L}_2$. Finally, z_1 and z_2 are the only two surfels of Σ which are e_x-adjacent to y.

Furthermore, we must state that $N_v(x)$ is e_x-connected. This comes immediately from the fact that $N_v(x)$ is made of the union of the four loops which contain x, minus the surfel x itself. Now, the loops can be ordered following the vertices order; each one is $e-$connected and shares a surfel with its successor in the latter order. It follows that the union (minus $\{x\}$) introduced before is e_x-connected. □

In order to illustrate the latter lemma, we have depicted in Figure 6.2 the e_x-adjacency graph G of $N_v(x)$ where x is a surfel the neighborhood of which is a simple closed e_x-curve but not a simple closed $e-$curve.

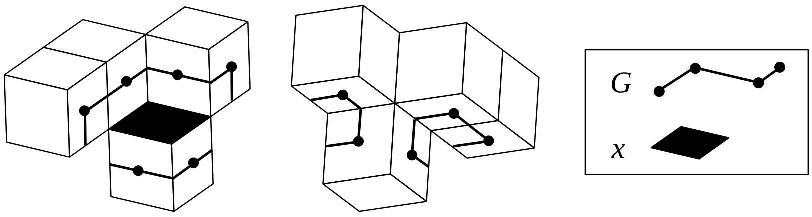


Figure 6.2: Front and back views of a surface $\Sigma = \delta_{18}(O, V)$, a surfel x (in black), and the graph G of e_x-adjacency in $N_v(x)$.

Now, given a surfel y in $N_v(x)$, there exists exactly two parameterizations (see Definition 1.13) $\pi = (y_k)_{k=0,\dots,p}$ and $\pi' = (y'_k)_{k=0,\dots,p'}$ of the simple closed e_x-curve $\mathcal{C} = N_v(x)$ such that $y_0 = y'_0$. Furthermore, is is immediate that $\pi^{-1} = \pi'$ (see Figure 6.3). Then, we can define as follows a canonical parameterization of the neighborhood of a surfel x which starts at a given surfel y of $N_v(x)$.

Definition 6.1 (canonical parameterization of $N_v(x)$) *Let $x \in \Sigma$ and $y \in N_v(x)$. We define the* canonical parameterization of $N_v(x)$ associated to the surfel y*, denoted by $\mathcal{C}_y(x)$, as the only e_x-path $\pi = (y_0, \dots, y_p)$ from $y = y_0$ to $y = y_p$ such that π*

is a parameterization of the simple closed e_x-*curve* $\mathcal{C} = N_v(x)$ *and which satisfies the following property : for all* $k \in \{0, \ldots, p-1\}$, $x \in \mathcal{L}_{w_k}(y_k)$ *where* $(w_k - 1 \bmod 4, w_k)$ *is the oriented edge of* y_k *shared as an edge by* y_k *and* y_{k+1}.

In other words, $\mathcal{C}_y(x)$ is the only e_x-path π from y to y in $N_v(x)$ such that x is always *on the left* of π for an observer which would walk on π and always look in the direction of the next surfel in the parameterization. In the case of Figure 6.3, we obtain that $\mathcal{C}_{y_0}(x)$ is the e_x-path π'. Indeed, arrows in the lower part of this figure depict the oriented edges of y_i [resp. y'_i] shared as edges by y_i and y_{i+1} [resp. y'_i and y'_{i+1}] for $i \in \{0, \ldots, 10\}$.

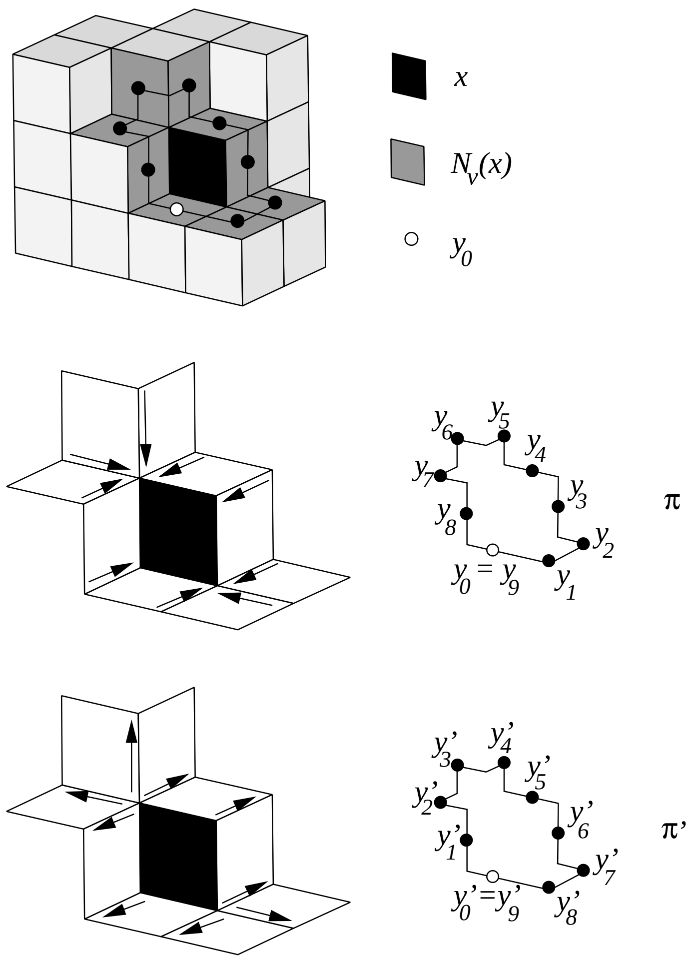


Figure 6.3: The two parameterizations $\pi = (y_0, \ldots, y_9)$ and $\pi' = (y'_0, \ldots, y'_9)$ of $N_v(x)$ from y to y.

We can now define locally, at each point y_k of an $n-$path π, the locals left and right sides of π on the surface, taking into account the orientation of the surface. These local left and right sets are subsets of the $v-$neighborhood of the surfel y_k.

Definition 6.2 (local left and right sets) *Let $\pi = (y_k)_{k=0,\ldots,p}$ be an $n-$path for $n \in \{e, v\}$ and $k \in \{1, \ldots, p-1\}$ ($k \in \{0, \ldots, p\}$ if π is closed). Then, let $\gamma = (\gamma_0, \ldots, \gamma_l) = \mathcal{C}_{y_{k-1}}(y_k)$ be the canonical parameterization of $N_v(y_k)$ associated to y_{k-1}. Let h be the only integer in $\{1, \ldots, l\}$ such that $y_{k+1} = \gamma_h$. We define the sets of surfels $Left_\pi(k)$ and $Right_\pi(k)$ by :*

If $l = h$ (i.e $y_{k-1} = y_{k+1}$) then $Left_\pi(k) = Right_\pi(k) = \emptyset$, otherwise,

$$\begin{aligned} Right_\pi(k) &= N_v(y_k) \cap \{\gamma_i \mid 0 < i < h-1\} \\ Left_\pi(k) &= N_v(y_k) \cap \{\gamma_i \mid h+1 < i < l\} \end{aligned}$$

Note that both sets $Right_\pi(0)$ and $Left_\pi(0)$ are not defined in the case when π is not closed (since the notation y_{i-1} has no meaning for $i = 0$ in this case).

A few examples of such sets $Left_\pi(k)$ and $Right_\pi(k)$ are depicted in Figure 6.5 for some $e-$paths and in Figure 6.6 for some $v-$paths which are not $e-$paths.

Observe that it is intuitively impossible to properly define the left and right sides of a walk when one only knows the previous and the next positions which are identical. However, we simply define these sets as empty sets in this case (see Figure 6.4).

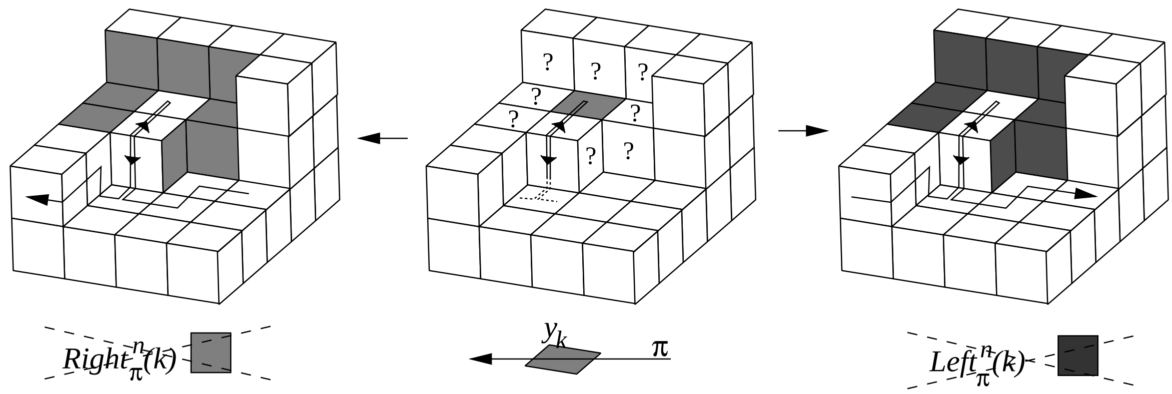


Figure 6.4: Illustration of the case when the local left an right sides are both defined as empty.

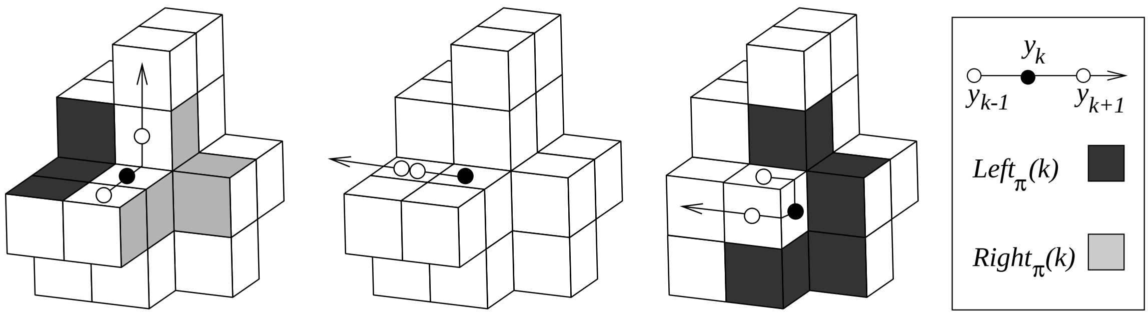
Now, the three following remarks are straightforward consequences of the latter definition.

Remark 6.1 *If $c = (x_i)_{i=0,\ldots,q}$ is an $n-$path [resp. a closed $n-$path] and $i \in \{1, \ldots, q-1\}$ [resp. $i \in \{0, \ldots, q\}$] is such that x_{i+1} and x_{i-1} are e$-$adjacent, then either $Right_c(i) = \emptyset$ and $Left_c(i) = N_v(x_i) \setminus \{x_{i-1}, x_{i+1}\}$; or $Left_c(i) = \emptyset$ and $Right_c(i) = N_v(x_i) \setminus \{x_{i-1}, x_{i+1}\}$. See Figure 6.5(c) for an example of such a situation. Conversely, one of these two sets may be empty only when the the two surfels x_{i-1} and x_{i+1} are either equal or e$-$adjacent.*

Remark 6.2 *If $c = (x_i)_{i=0,\ldots,q}$ is an $n-$path on Σ, then $Left_c(i) \cap Right_c(i) = \emptyset$ for all $i \in \{1, \ldots, q-1\}$ (for all $i \in \{0, \ldots, q-1\}$ if c is closed).*

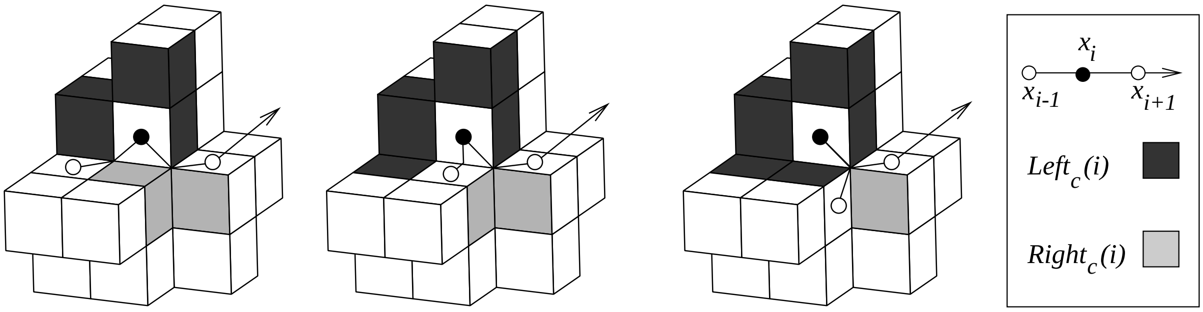
Remark 6.3 *If $c = (x_i)_{i=0,\ldots,q}$ is an $n-$path on Σ and x_i is a surfel of c such that x_{i-1} and x_{i+1} are neither equal nor $n_{x_i}-$adjacent; then the sets $Left_c(i)$ and $Right_c(i)$ are both non empty and each contains a surfel which is $\overline{n}-$adjacent to x_i.*

Two examples of configurations which satisfy the latter remark are depicted in Figure 6.5(a) and Figure 6.6(a). Indeed, the two surfels x_{i-1} and x_{i+1} of Figure 6.6(a) are not v_x-adjacent so that $Left_c(i) \cap N_e(x) \neq \emptyset$ and $Right_c(i) \cap N_e(x) \neq \emptyset$. Finally, some counter examples are depicted in Figure 6.5(c) and Figure 6.6(b).



(a) (b) $Left_\pi(k) = \emptyset$. (c) $Left_\pi(k) = \emptyset$.

Figure 6.5: Some illustrations of the sets $Left_\pi(k)$ and $Right_\pi(k)$ where $\pi = (y_k)_{k=0,\ldots,p}$.



(a) (b) $Right_c(i) \cap N_e(x) = \emptyset$. (c) $N_e(x) \subset Left_c(i)$ and $Right_c(i) \cap N_e(x) = \emptyset$.

Figure 6.6: Some illustrations of the sets $Left_c(i)$ and $Right_c(i)$ where $c = (x_i)_{i=0,\ldots,q}$ is a $v-$path which is not an $e-$path.

The following property is the necessary and sufficient condition which will allow the definition of the *intersection number* between two paths.

Notation 6.3 Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path in Σ. We say that the property $\mathcal{P}(\pi, c)$ is satisfied if in case when π is not closed then neither y_0 nor y_p belongs to c^*.

Now we define the contribution to the intersection number of a couple of subscripts.

Definition 6.3 (contribution to the intersection number) *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path such that $\mathcal{P}(\pi, c)$ holds. Let $k \in \{0, \dots, p-1\}$ and $i \in \{0, \dots, q\}$. We define the* contribution to the intersection number *of the couple (k, i) denoted by $\mathcal{I}_{\pi,c}(k, i)$ which is equal to zero if $x_i \neq y_k$, otherwise $\mathcal{I}_{\pi,c}(k, i) = \mathcal{I}^-_{\pi,c}(k, i) + \mathcal{I}^+_{\pi,c}(k, i)$ where :*

$\mathcal{I}^-_{\pi,c}(k, i) = 0$ *if* $i = 0$,	$\mathcal{I}^+_{\pi,c}(k, i) = 0$ *if* $i = q$,
$\mathcal{I}^-_{\pi,c}(k, i) = 0.5$ *if* $x_{i-1} \in Right_\pi(k)$,	$\mathcal{I}^+_{\pi,c}(k, i) = -0.5$ *if* $x_{i+1} \in Right_\pi(k)$,
$\mathcal{I}^-_{\pi,c}(k, i) = -0.5$ *if* $x_{i-1} \in Left_\pi(k)$,	$\mathcal{I}^+_{\pi,c}(k, i) = 0.5$ *if* $x_{i+1} \in Left_\pi(k)$,
$\mathcal{I}^-_{\pi,c}(k, i) = 0$ *in all other cases.*	$\mathcal{I}^+_{\pi,c}(k, i) = 0$ *in all other cases.*

Note that $\mathcal{I}_{\pi,c}(k, i) = 0$ if $x_{i-1} = x_{i+1}$ or $y_{k-1} = y_{k+1}$ (since $Left_\pi(k) = Right_\pi(k) = \emptyset$ in this case).

Note that $\mathcal{I}^-_{\pi,c}(k, i)$ depends on the position of x_{i-1} relative to the $n-$path π at the surfel y_k, and $\mathcal{I}^+_{\pi,c}(k, i)$ depends on the position of x_{i+1}. Also observe that $\mathcal{I}_{\pi,c}(0, i) = 0$ for all $i \in \{0, \dots, q\}$ if π is not closed since $\mathcal{P}(\pi, c)$ implies that $x_i \neq y_0$ for all $i \in \{0, \dots, q\}$ in this case. Indeed, otherwise $\mathcal{I}_{\pi,c}(0, i)$ would not be defined when π is not closed and $x_i = y_0$.

Definition 6.4 (Intersection Number) *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and let $c = (x_i)_{i=0,\dots,q}$ be a $\overline{n}-$path such that the property $\mathcal{P}(\pi, c)$ holds. The* intersection number *of the $n-$path π and the $\overline{n}-$path c, denoted by $\mathcal{I}_{\pi,c}$, is defined by :*

$$\mathcal{I}_{\pi,c} = \sum_{k=0}^{p-1} \sum_{i=0}^{q} \mathcal{I}_{\pi,c}(k, i) = \sum_{k=0}^{p-1} \sum_{i|x_i=y_k} \mathcal{I}_{\pi,c}(k, i) = \sum_{i=0}^{q} \sum_{k|x_i=y_k} \mathcal{I}_{\pi,c}(k, i).$$

Notation 6.4 Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path such that $\mathcal{P}(\pi, c)$ holds, then, for $h \in \{0, \dots, p\}$ and $l \in \{0, \dots, q\}$ we denote :

$$\mathcal{I}^\pi_{\pi,c}(l) = \sum_{k=0}^{p-1} \mathcal{I}_{\pi,c}(k, l) \text{ and } \mathcal{I}^c_{\pi,c}(h) = \sum_{i=0}^{q} \mathcal{I}_{\pi,c}(h, i).$$

We have depicted in Figure 6.7 and Figure 6.8 two examples of pairs of paths. Although any closed path in the digital surface of Figure 6.7 is reducible in the whole surface, this is not true in the surface of Figure 6.8. Indeed, the two paths drawn in this figure have an intersection number of ± 1 depending on their parameterization, and this will be sufficient (using the properties which will be stated in the sequel) to prove that they are not reducible.

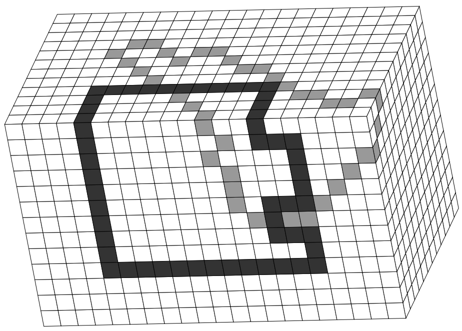


Figure 6.7: A $v-$path c (in grey) and an $e-$path π (in black) such that $\mathcal{I}_{\pi,c} = 0$.

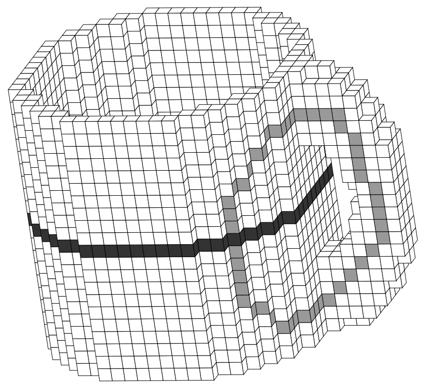


Figure 6.8: A $v-$path c and an $e-$path π such that $\mathcal{I}_{\pi,c} = \pm 1$.

Remark 6.4 *The intersection number $\mathcal{I}_{\pi,c}$ is not defined when the path $\pi = (y_k)_{k=0,\ldots,p}$ is not closed and some surfels of c belong to $\{y_0, y_p\}$ (i.e. when $\mathcal{P}(\pi, c)$ is not satisfied). However, $\mathcal{I}_{\pi,c}$ is well defined in the case when $c = (x_i)_{i=0,\ldots,q}$ is not closed and π^* contains one or both of x_0 and x_q. In the latter case, $\mathcal{I}_{\pi,c}$ may not be an integer (see Figure 6.9).*

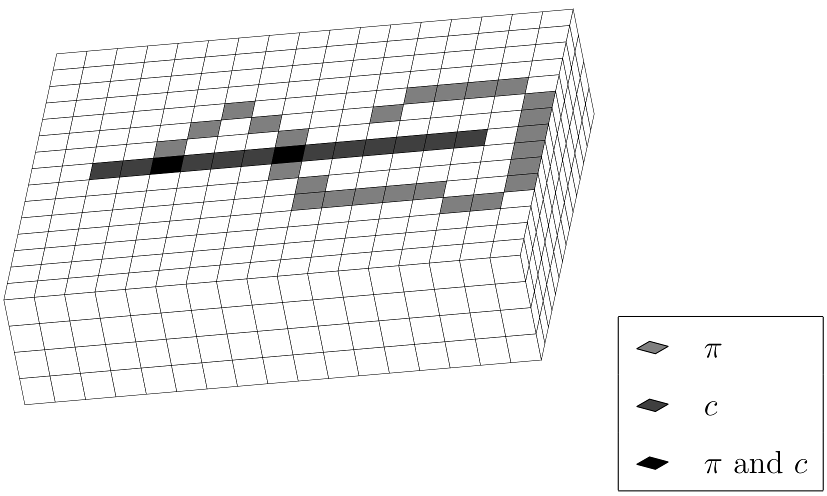


Figure 6.9: The intersection $\mathcal{I}_{c,\pi}$ is not defined since π^* contains one extremity of c. However, $\mathcal{I}_{\pi,c} = \pm 0.5$.

Remark 6.5 *From the very definition of $\mathcal{I}_{\pi,c}$, we have $\mathcal{I}_{\pi,c} = 0$ as soon as π or c is a trivial path (Definition 1.12).*

In further proof, we will use the following Definition.

Definition 6.5 *Let π be an $n-$path in Σ and $\pi' = (y'_0, y'_1)$ be an $\overline{n}-$path in Σ with a length 1. We say that π'* enters *π if $y'_0 \notin \pi^*$ and $y'_1 \in \pi^*$; and we say that π'* exits *π if $y'_0 \in \pi^*$ and $y'_1 \notin \pi^*$.*

6.2 Main Properties

In this section, we introduce the main theorems relative to the intersection number which was first stated in [32] and [30] with a less comprehensive proof. Indeed, the proofs which will be given here are more concise then the ones in previously mentioned papers.

Theorem 9 *Let $\pi = (y_k)_{k=0,\ldots,p}$ be an $n-$path in Σ ($n \in \{e, v\}$). Furthermore, let $c = (x_i)_{i=0,\ldots,q}$ and $c' = (x'_i)_{i=0,\ldots,q'}$ be two $\overline{n}-$paths such that $\mathcal{P}(\pi, c)$ and $\mathcal{P}(\pi, c')$ hold. If c' is $\overline{n}-$homotopic to c in Σ (in $\Sigma \setminus \{y_0, y_p\}$ if π is not closed), then $\mathcal{I}_{\pi,c} = \mathcal{I}_{\pi,c'}$.*

In other words, the intersection number between an $n-$path π and an $\overline{n}-$path c, as defined in previous subsection, is invariant under any homotopic deformation applied to the path c. First, we prove the following Proposition which states that the intersection number has a commutative property (up to a change of sign).

Proposition 6.2 *Let $\pi = (y_k)_{k=0,\ldots,p}$ be an $n-$path of Σ and $c = (x_i)_{i=0,\ldots,q}$ be an $\overline{n}-$path of Σ such that both $\mathcal{P}(\pi, c)$ and $\mathcal{P}(c, \pi)$ hold. Then, $\mathcal{I}_{\pi,c} = -\mathcal{I}_{c,\pi}$.*

The Property stated by Theorem 9 can be used together with Proposition 6.2 to show that a closed $n-$path α ($n \in \{e, v\}$) is not $n-$homotopic to a trivial path by finding an $\overline{n}-$path β whose intersection number with α is not equal to zero. More generally, it can be used to distinguish two not $n-$homotopic paths if their intersection numbers with a third $\overline{n}-$path are different.

Indeed, the following theorem is an immediate consequence of both Theorem 9 and Proposition 6.2.

Theorem 10 *Let $\pi = (y_k)_{k=0,\ldots,p}$ and $\pi' = (y'_k)_{k=0,\ldots,p'}$ be two $n-$paths in Σ ($n \in \{e, v\}$). Furthermore, let $c = (x_i)_{i=0,\ldots,q}$ be an $\overline{n}-$path such that the properties $\mathcal{P}(\pi, c)$, $\mathcal{P}(\pi', c)$, $\mathcal{P}(c, \pi)$, and $\mathcal{P}(c, \pi')$ hold. If π' is $n-$homotopic to π in Σ (in $\Sigma \setminus \{x_0, x_q\}$ if c is not closed), then $\mathcal{I}_{\pi,c} = \mathcal{I}_{\pi',c}$.*

Proof : From Proposition 6.2 and since $\mathcal{P}(\pi, c)$ and $\mathcal{P}(c, \pi)$ hold we have $\mathcal{I}_{\pi,c} = -\mathcal{I}_{c,\pi}$. On the other hand, still from Proposition 6.2 and since $\mathcal{P}(\pi', c)$ and $\mathcal{P}(c, \pi')$ hold we have $\mathcal{I}_{\pi',c} = -\mathcal{I}_{c,\pi'}$. Finally, from Theorem 9, since $\mathcal{P}(c, \pi)$ and $\mathcal{P}(c, \pi')$ hold; and since π' is $n-$homotopic to π in Σ (in $\Sigma \setminus \{x_0, x_q\}$ if c is not closed) then $\mathcal{I}_{c,\pi} = \mathcal{I}_{c,\pi'}$. □

The proof of Theorem 9 will come after the following section which states several useful properties of the intersection number.

6.3 Useful Properties

6.3.1 Change of sign with path inversion

Proposition 6.3 *Let π be an $n-$path and c be an $\overline{n}-$path such that $\mathcal{P}(\pi, c)$ holds. Then $\mathcal{I}_{\pi,c} = -\mathcal{I}_{\pi^{-1},c}$.*

In order to prove Proposition 6.3, we first state the following Lemmas.

Lemma 6.4 *Let $\pi = (y_k)_{k=0,\ldots,p}$ be an $n-$path in Σ. Then, $Left_\pi(k) = Right_{\pi^{-1}}(p-k)$ and $Right_\pi(k) = Left_{\pi^{-1}}(p-k)$ for all $k \in \{1, \ldots, p-1\}$. Furthermore, if π is closed, then $Left_\pi(0) = Right_{\pi^{-1}}(0)$ and $Right_\pi(0) = Left_{\pi^{-1}}(0)$.*

Proof : Let $\pi = (y_k)_{k=0,\ldots,p}$ and $\pi^{-1} = (y'_k)_{k=0,\ldots,p}$.

If π is closed, then $y_0 = y'_0$, $y_1 = y'_{p-1}$ and $y_{p-1} = y'_1$. Let $\beta = \mathcal{C}_{y_{p-1}}(y_0) = (\beta^0, \ldots, \beta^{l_0})$ be the canonical parameterization of $N_v(y_0)$ associated to y_{p-1}. And let h_0 be the only integer of $\{1, \ldots, l_0\}$ such that $y_{p-1} = \beta^{h_0}$. If $h_0 = l_0$ it is immediate that $Left_\pi(0) = Right_{\pi^{-1}}(0) = Right_\pi(0) = Left_{\pi^{-1}}(0) = \emptyset$. If $h_0 < l_0$ then it is also immediate that $\beta' = (\beta^{h_0}, \beta^{h_0+1}, \ldots, \beta^{l_0}).(\beta^0, \beta^1, \ldots, \beta^{h_0})$ is the canonical parameterization of $N_v(y'_0) = N_v(y_0)$ associated to the surfel $y'_{p-1} = y_1$ (see Definition 6.1). Finally, from Definition 6.2, $Left_\pi(0) = Right_{\pi^{-1}}(0)$ and $Right_\pi(0) = Left_{\pi^{-1}}(0)$.

Now, for all $k \in \{1, \ldots, p-1\}$ we observe that $y_k = y'_{p-k}$, $y_{k-1} = y'_{(p-k)+1}$ and $y_{k+1} = y'_{(p-k)-1}$. For such k, let $\gamma = \mathcal{C}_{y_{k-1}}(y_k) = (\gamma^0, \ldots, \gamma^l)$ be the canonical parameterization of $N_v(y_k)$ associated to y_{k-1}. And let h be the only integer of $\{1, \ldots, l\}$ such that $y_{k+1} = \gamma^h$. If $h = l$ it is immediate that $Left_\pi(k) = Right_{\pi^{-1}}(p-k) = Right_\pi(k) = Left_{\pi^{-1}}(p-k) = \emptyset$. If $h < l$ then it is also immediate that $\gamma' = (\gamma^h, \gamma^{h+1}, \ldots, \gamma^l).(\gamma^0, \gamma^1, \ldots, \gamma^h)$ is the canonical parameterization of $N_v y'_{p-k} = N_v y_k$ associated to the surfel $y'_{(p-k)-1} = y_{k+1}$ (see

Definition 6.1). Finally, from Definition 6.2, $Left_\pi(k) = Right_{\pi^{-1}}(p-k)$ and $Right_\pi(k) = Left_{\pi^{-1}}(p-k)$. $\square$

Lemma 6.5 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path with a length p in Σ and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path with a length q in Σ such that $\mathcal{P}(\pi, c)$ holds. Then, $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi^{-1},c}(p-k,i)$ for all $k \in \{1,\dots,p-1\}$ and all $i \in \{0,\dots,q\}$. If π is closed, then $\mathcal{I}_{\pi,c}(0,i) = -\mathcal{I}_{\pi^{-1},c}(0,i)$ for all $i \in \{0,\dots,q\}$.*

Proof : Let $\pi^{-1} = (y'_0,\dots,y'_p)$. From Lemma 6.5, $Right_\pi(k) = Left_{\pi^{-1}}(p-k)$ and $Left_\pi(k) = Right_{\pi^{-1}}(p-k)$ for all $k \in \{1,\dots,p-1\}$. Then, following Definition 6.3, we have $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi^{-1},c}(p-k,i)$ for all $i \in \{0,\dots,q\}$. If π is closed and still from Lemma 6.4 and Definition 6.3, we have $Right_\pi(0) = Left_{\pi^{-1}}(0)$ and $Left_\pi(0) = Right_{\pi^{-1}}(0)$ so that $\mathcal{I}_{\pi,c}(0,i) = -\mathcal{I}_{\pi^{-1},c}(0,i)$ for all $i \in \{0,\dots,q\}$. $\square$

Proof of Proposition 6.3 : Let $\pi = (y_0,\dots,y_p)$, $\pi^{-1} = (y'_0,\dots,y'_p)$ and $c = (x_0,\dots,x_q)$.

$$\mathcal{I}_{\pi,c} = \left[\sum_{i=0}^{q} \mathcal{I}_{\pi,c}(0,i)\right] + \sum_{k=1}^{p-1}\sum_{i=0}^{q} \mathcal{I}_{\pi,c}(k,i) \tag{6.1}$$

$$\mathcal{I}_{\pi^{-1},c} = \left[\sum_{i=0}^{q} \mathcal{I}_{\pi^{-1},c}(0,i)\right] + \sum_{k=1}^{p-1}\sum_{i=0}^{q} \mathcal{I}_{\pi^{-1},c}(p-k,i) \tag{6.2}$$

Following Lemma 6.5, we have $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi^{-1},c}(p-k,i)$ for all $k \in \{1,\dots,p-1\}$ and all $i \in \{0,\dots,q\}$. Furthermore, if π is not closed and since $\mathcal{P}(\pi,c)$ holds, then $\mathcal{I}_{\pi,c}(0,i) = \mathcal{I}_{\pi^{-1},c}(0,i) = 0$ for all $i \in \{0,\dots,q\}$ (since $x_i \neq y_0$ for such i). If π is closed and still from Lemma 6.5, we have $\mathcal{I}_{\pi,c}(0,i) = -\mathcal{I}_{\pi^{-1},c}(0,i)$ for all $i \in \{0,\dots,q\}$. Finally, $\mathcal{I}_{\pi,c} = -\mathcal{I}_{\pi^{-1},c}$ from equations (6.1) and (6.2). $\square$

Proposition 6.6 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path such that $\mathcal{P}(\pi,c)$ holds; then $\mathcal{I}_{\pi,c} = -\mathcal{I}_{\pi,c^{-1}}$.*

In order to prove Proposition 6.6, we first establish the following Lemma.

Lemma 6.7 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path with a length p in Σ and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path with a length q in Σ such that $\mathcal{P}(\pi,c)$ holds. Then, $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi,c^{-1}}(k,q-i)$ for all $k \in \{0,\dots,p-1\}$ and all $i \in \{0,\dots,q\}$.*

Proof : Let $c^{-1} = (x'_0, \dots, x'_q)$ so that for all $i \in \{0, \dots, q\}$ we have $x_i = x'_{q-i}$.

• For $i \in \{1, \dots, q-1\}$ we observe that $x_i = x'_{q-i}$, $x_{i-1} = x'_{(q-i)+1}$, $x_{i+1} = x'_{(q-i)-1}$. Thus from Definition 6.3 and for such i, we have $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi,c^{-1}}(k, q-i)$ for all $k \in \{0, \dots, p-1\}$.

• For $i = 0$, since $x_0 = x'_q$ and $x_1 = x'_{q-1}$ we also have $\mathcal{I}_{\pi,c}(k,0) = \mathcal{I}^+_{\pi,c}(k,0) = -\mathcal{I}^-_{\pi,c^{-1}}(k,q) = -\mathcal{I}_{\pi,c^{-1}}(k, q-0)$ for all $k \in \{0, \dots, p-1\}$.

• For $i = q$, since $x_q = x'_0$ and $x_{q-1} = x'_1$ we also have $\mathcal{I}_{\pi,c}(k,q) = \mathcal{I}^-_{\pi,c}(k,q) = -\mathcal{I}^+_{\pi,c^{-1}}(k,0) = -\mathcal{I}_{\pi,c^{-1}}(k, q-q)$ for all $k \in \{0, \dots, p-1\}$.

Finally, for all $k \in \{0, \dots, p-1\}$ and all $i \in \{1, \dots, q-1\}$ we have $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi,c}(k, q-i)$. $\square$

Proof of Proposition 6.6 : Let $c^{-1} = (x'_0, \dots, x'_q)$ so that for all $i \in \{0, \dots, q\}$ we have $x_i = x'_{q-i}$. Then,

$$\mathcal{I}_{\pi,c} = \sum_{k=0}^{p-1} \sum_{i=0}^{q} \mathcal{I}_{\pi,c}(k,i)$$

But, from Lemma 6.7, $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi,c^{-1}}(k, q-i)$ for all $i \in \{0, \dots, q\}$ and all $k \in \{0, \dots, p-1\}$. It is then immediate that,

$$\mathcal{I}_{\pi,c} = \sum_{k=0}^{p-1} \sum_{i=0}^{q} -\mathcal{I}_{\pi,c^{-1}}(k,i) = -\mathcal{I}_{\pi,c^{-1}}$$

$\square$

6.3.2 Commutativity property

In further proofs, we will use Proposition 6.2 which was introduced in Section 6.2 and which sates that swapping the roles played by the two paths in the definition of the intersection number leads to a change of the sign of this intersection number, when such a permutation is possible. Indeed, in the case when π is closed and c is not closed, then if an extremity of c belongs to π^* the intersection number $\mathcal{I}_{\pi,c}$ is well defined whereas the number $\mathcal{I}_{c,\pi}$ is not. The idea of this commutativity property is summarized in Figure 6.10 where one can say that c crosses π from left to right by observing one of the two following statements :

– c enters π from the left side at the point a and exits π to the right side of π at the point b, or

– π enters c from the right side of c at the point a and exits c to the left side of c at the point b.

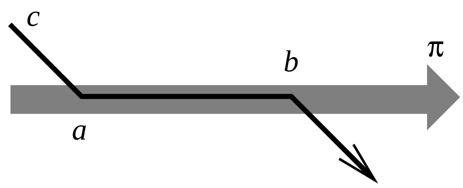


Figure 6.10: There are two ways to check that c crosses π from left to right.

We recall Proposition 6.2 :

Proposition 6.2 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path of Σ and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path of Σ such that both $\mathcal{P}(\pi, c)$ and $\mathcal{P}(c, \pi)$ hold. Then, $\mathcal{I}_{\pi,c} = -\mathcal{I}_{c,\pi}$.*

In order to prove this very intuitive result, we must state several technical lemmas.

Lemma 6.8 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path of Σ and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path of Σ such that both $\mathcal{P}(\pi, c)$ and $\mathcal{P}(c, \pi)$ hold. For all $k \in \{1, \dots, p-1\}$ ($k \in \{0, \dots, p-1\}$ if π is closed) and all $i \in \{1, \dots, q-1\}$, we have $\mathcal{I}_{\pi,c}(k, i) = -\mathcal{I}_{c,\pi}(i, k)$.*

Proof of Lemma 6.8 : Following Definition 6.3, the cases when $x_i \neq y_k$, $x_{i-1} = x_{i+1}$ or $y_{k-1} = y_{k+1}$ are immediate since in these cases $\mathcal{I}_{\pi,c}(k, i) = \mathcal{I}_{c,\pi}(i, k) = 0$. Thus, we suppose in the sequel of this proof that $x_i = y_k$, $x_{i-1} \neq x_{i+1}$, $y_{k-1} \neq y_{k+1}$. By the same way, if $x_{i-1} = y_{k-1}$ and $x_{i+1} = y_{k+1}$, or if $x_{i-1} = y_{k+1}$ and $x_{i+1} = y_{k-1}$ it is immediate that $\mathcal{I}_{\pi,c}(k, i) = \mathcal{I}_{c,\pi}(i, k) = 0$. Then we may also suppose that $\{y_{k-1}, y_{k+1}\} \neq \{x_{i-1}, x_{i+1}\}$; the following cases remain :

Case 1 $x_{i-1} = y_{k-1}$ (see Figure 6.11) so that $\mathcal{I}^-_{\pi,c}(k, i) = \mathcal{I}^-_{c,\pi}(i, k) = 0$. Then, let $\gamma = (\gamma^0, \dots, \gamma^l) = \mathcal{C}_{y_{k-1}}(y_k) = \mathcal{C}_{x_{i-1}}(x_i)$ be the canonical parameterization of $N_v(y_k) = N_v(x_i)$ and h be the only integer in $\{1, \dots, l\}$ such that $y_{k+1} = \gamma^h$. Following assumptions made before, $x_{i+1} \notin \{\gamma^0 = \gamma^l, \gamma^h\}$.

If $x_{i+1} = \gamma^j$ for $0 < j < h$ then $x_{i+1} \in Right_\pi(k)$ and $h > j$ implies that $y_{k+1} \in Left_c(i)$ (see Definition 6.2). From Definition 6.3, it follows that $\mathcal{I}^+_{\pi,c}(k, i) = -\mathcal{I}^+_{c,\pi}(i, k) = 0.5$ and finally $\mathcal{I}_{\pi,c}(k, i) = -\mathcal{I}_{c,\pi}(i, k)$.

If $x_{i+1} = \gamma^j$ for $h < j < l$ then $x_{i+1} \in Left_\pi(k)$ and $j > h$ implies that $y_{k+1} \in Right_c(i)$ (see Definition 6.2). From Definition 6.3, it follows that $\mathcal{I}^+_{\pi,c}(k, i) = -\mathcal{I}^+_{c,\pi}(i, k) = 0.5$ so that $\mathcal{I}_{\pi,c}(k, i) = -\mathcal{I}_{c,\pi}(i, k)$.

Case 2 $x_{i-1} = y_{k+1}$ (see Figure 6.11) so that $\mathcal{I}^-_{\pi,c}(k,i) = \mathcal{I}^-_{c,\pi}(i,k) = 0$. We observe that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi^{-1},c}(p-k,i)$ and $\mathcal{I}_{c,\pi}(i,k) = -\mathcal{I}_{c,\pi^{-1}}(i,p-k)$ (Lemma 6.5). Thus, we must prove that $\mathcal{I}_{\pi^{-1},c}(p-k,i) = \mathcal{I}_{c,\pi^{-1}}(i,p-k)$. Now, let $\pi^{-1} = (y'_0,\dots,y'_p)$ then $y_k = y'_{p-k}$ and $y_{k+1} = y'_{(p-k)-1}$ and we are came down to the previous case at subscript $p-k$ of π^{-1}.

Case 3 $x_{i+1} = y_{k+1}$ (see Figure 6.11) so that $\mathcal{I}^+_{\pi,c}(k,i) = \mathcal{I}^+_{c,\pi}(i,k) = 0$. We observe that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi^{-1},c}(p-k,i) = \mathcal{I}_{\pi^{-1},c^{-1}}(p-k,q-i)$ (from Lemma 6.5 and Lemma 6.7), by the same way $\mathcal{I}_{c,\pi}(i,k) = \mathcal{I}_{c^{-1},\pi^{-1}}(q-i,p-k)$. It is then sufficient to prove that $\mathcal{I}_{\pi^{-1},c^{-1}}(p-k,q-i) = -\mathcal{I}_{c^{-1},\pi^{-1}}(p-k,q-i)$. If $c^{-1} = (x'_0,\dots,x'_q)$ and $\pi^{-1} = (y'_0,\dots,y'_q)$ then, on a first hand $x_i = x'_{q-i}$, $x_{i+1} = x'_{(q-i)-1}$, $x_{i-1} = x'_{(q-i)+1}$. On the other hand, $y_k = y'_{p-k \bmod p}$, $y_{k-1 \bmod p} = y'_{(p-k)+1 \bmod p}$ and $y_{k+1 \bmod p} = y'_{(p-k)-1 \bmod p}$. We are came back to case 1 with the subscripts $(p-k \bmod p)$ and $(q-i \bmod p)$ so that this cases is equivalent to case 1.

Case 4 $x_{i+1} = y_{k-1}$ (see Figure 6.11) so that $\mathcal{I}^+_{\pi,c}(k,i) = \mathcal{I}^+_{c,\pi}(i,k) = 0$. We observe that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{\pi,c^{-1}}(k,q-i)$ and $\mathcal{I}_{\pi,c}(c,\pi) = -\mathcal{I}_{c^{-1},\pi}(q-i,k)$ (Lemma 6.7). It is then sufficient to prove that $\mathcal{I}_{\pi,c^{-1}}(k,q-i) = \mathcal{I}_{c^{-1},\pi}(q-i,k)$. If $c^{-1} = (x'_0,\dots,x'_q)$ then $x_i = x'_{q-i}$, $x_{i-1} = x'_{(q-i)+1}$ and $x_{i+1} = x'_{(q-i)-1}$ so that this cases is equivalent to case 1.

Case 5 If $\{x_{i+1}, x_{i-1}\} \cap \{y_{k-1}, y_{k+1}\} = \emptyset$.

Let $\gamma = (\gamma^0,\dots,\gamma^l) = \mathcal{C}_{y_{k-1}}(y_k)$ be the canonical parameterization of $N_v(y_k)$ and h be the only integer in $\{1,\dots,l\}$ such that $y_{k+1} = \gamma^h$. Then there exists m and m' such that $\gamma^m = x_{i-1}$ and $\gamma^{m'} = x_{i+1}$. Since, $\{x_{i+1}, x_{i-1}\} \cap \{\gamma^0 = \gamma^l, \gamma^h\} = \emptyset$ it follows that $\{m, m'\} \subset \{1,\dots,h-1\} \cup \{h+1,\dots,l-1\}$. The following cases are illustrated in Figure 6.12.

$i)$ If $1 < m < m' < h$ then

$$\gamma' = (\gamma^m,\dots,\gamma^{m'}).(\gamma^{m'},\dots,\gamma^h).(\gamma^h,\dots,\gamma^l).(\gamma^0,\dots,\gamma^m)$$

is the canonical parameterization $\mathcal{C}_{x_{i-1}}(x_i)$ of $N_v(x_i)$. It follows that $\{x_{i-1}, x_{i+1}\} \subset \mathit{Right}_\pi(k)$ and $\{\gamma^0 = y_{k-1}, \gamma^h = k_{k+1}\} \subset \mathit{Left}_c(i)$. Finally, from Definition 6.3, we obtain that $\mathcal{I}_{\pi,c}(k,i) = \mathcal{I}_{c,\pi}(i,k) = 0$.

ii) If $1 < m' < m < h$ then

$$\gamma' = (\gamma^m, \ldots, \gamma^h).(\gamma^h, \ldots, \gamma^l).(\gamma^0, \ldots, \gamma^{m'}).(\gamma^{m'}, \ldots, \gamma^m)$$

is the canonical parameterization $\mathcal{C}_{x_{i-1}}(x_i)$ of $N_v(x_i)$. It follows that $\{x_{i-1}, = x_{i+1}\} \subset Right_\pi(k)$ and $\{\gamma^l = y_{k-1}, \gamma^h = k_{k+1}\} \subset Left_c(i)$. Finally, from Definition 6.3, we obtain that $\mathcal{I}_{\pi,c}(k,i) = \mathcal{I}_{c,\pi}(i,k) = 0$.

iii) If $h < m < m' < l$ then

$$\gamma' = (\gamma^m, \ldots, \gamma^{m'}).(\gamma^{m'}, \ldots, \gamma^l).(\gamma^0, \ldots, \gamma^h).(\gamma^h, \ldots, \gamma^m)$$

is the canonical parameterization $\mathcal{C}_{x_{i-1}}(x_i)$ of $N_v(x_i)$. It follows that $\{x_{i-1}, x_{i+1}\} \subset Left_\pi(k)$ and $\{\gamma^0 = y_{k-1}, \gamma^h = k_{k+1}\} \subset Left_c(i)$. Finally, from Definition 6.3, we obtain that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k) = 0$.

iv) If $h < m' < m < l$ then

$$\gamma' = (\gamma^m, \ldots, \gamma^l).(\gamma^0, \ldots, \gamma^h).(\gamma^h, \ldots, \gamma^{m'}).(\gamma^{m'}, \ldots, \gamma^m)$$

is the canonical parameterization $\mathcal{C}_{x_{i-1}}(x_i)$ of $N_v(x_i)$. It follows that $\{x_{i-1}, x_{i+1}\} \subset Left_\pi(k)$ and $\{\gamma^l = y_{k-1}, \gamma^h = k_{k+1}\} \subset Right_c(i)$. Finally, from Definition 6.3, we obtain that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k) = 0$.

v) If $0 < m < h < m' < l$ then

$$\gamma' = (\gamma^m, \ldots, \gamma^h).(\gamma^h, \ldots, \gamma^{m'}).(\gamma^{m'}, \ldots, \gamma^l).(\gamma^0, \ldots, \gamma^m)$$

is the canonical parameterization $\mathcal{C}_{x_{i-1}}(x_i)$ of $N_v(x_i)$. It is then straightforward that $x_{i-1} \in Right_\pi(k)$, $x_{i+1} \in Left_\pi(k)$, $y_{k-1} = \gamma^0 \in Left_c(i)$ and $y_{k+1} = \gamma^h \in Right_c(i)$. Finally, from Definition 6.3, we obtain that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k) = +1$.

vi) If $0 < m' < h < m < l$ then

$$\gamma' = (\gamma^m, \ldots, \gamma^l).(\gamma^0, \ldots, \gamma^{m'}).(\gamma^{m'}, \ldots, \gamma^h).(\gamma^h, \ldots, \gamma^m)$$

is the canonical parameterization $\mathcal{C}_{x_{i-1}}(x_i)$ of $N_v(x_i)$. It follows that $x_{i-1} \in Left_\pi(k)$, $x_{i+1} \in Right_\pi(k)$, $y_{k-1} = \gamma^0 \in Right_c(i)$ and $y_{k+1} = \gamma^h \in Left_c(i)$. Finally, from Definition 6.3, we obtain that $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k) = -1$.

□

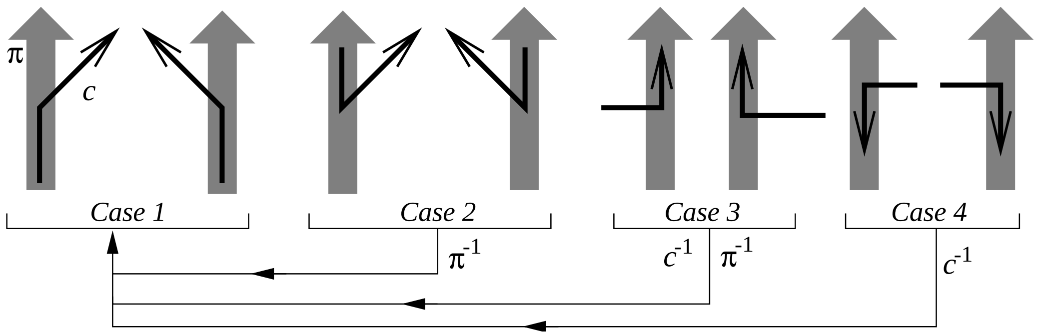


Figure 6.11: An illustration of the cases investigated in the proof of Lemma 6.8 when $\{x_{i-1}, x_{i+1}\} \cap \{y_{k-1}, y_{k+1}\} \neq \emptyset$

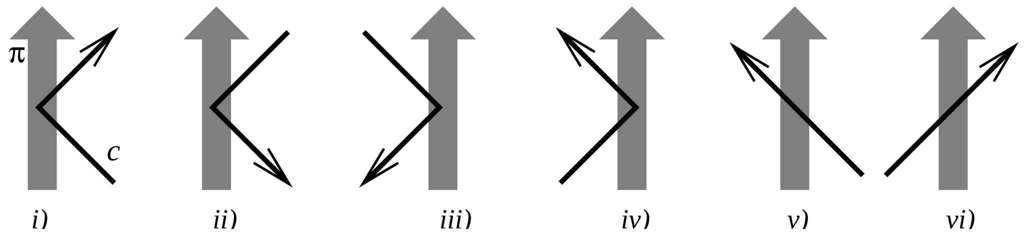


Figure 6.12: An illustration of the cases investigated in the proof of Lemma 6.8 when $\{x_{i-1}, x_{i+1}\} \cap \{y_{k-1}, y_{k+1}\} = \emptyset$

The following definition will allow us to use Lemma 6.8 for closed paths at subscripts corresponding to the extremities of either the path c or the path π of this lemma when the path is closed.

Definition 6.6 (shift operation) *Let $\pi = (y_k)_{k=0,\ldots,p}$ be a closed $n-$path in Σ with a length $p > 1$. We denote by $Sh(\pi)$ the closed $n-$path $(y_{p-1}, y_0, \ldots, y_{p-1})$ which is the result of a shift of π of one step in the opposite direction of its parameterization.*

Then, the two following Lemmas will be of interest in the sequel.

Lemma 6.9 *Let $\pi = (y_k)_{k=0,\ldots,p}$ be a closed $n-$path and $c = (x_i)_{i=0,\ldots,q}$ be an $\overline{n}-$path in Σ. If π has a length $p > 1$, then $\mathcal{I}_{\pi,c}(0, i) = \mathcal{I}_{Sh(\pi),c}(1, i)$ for all $i \in \{0, \ldots, q\}$.*

Corollary 6.10 *Let $\pi = (y_k)_{k=0,\ldots,p}$ be a closed $n-$path and $c = (x_i)_{i=0,\ldots,q}$ be an $\overline{n}-$path in Σ, then $\mathcal{I}_{\pi,c} = \mathcal{I}_{Sh(\pi),c}$.*

Proof of Lemma 6.9 : Let $Sh(\pi) = (y'_0, \ldots, y'_p)$ so that $y_{p-1} = y'_0$, $y_0 = y'_1$ and $y_1 = y'_2$. It follows that $Right_\pi(0) = Right_{Sh(\pi)}(1)$ and $Left_\pi(0) = Left_{Sh(\pi)}(1)$ so that $\mathcal{I}_{\pi,c}(0, i) = \mathcal{I}_{Sh(\pi),c}(1, i)$ for all $i \in \{0, \ldots, q\}$. $\square$

Lemma 6.11 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and $c = (x_i)_{i=0,\dots,q}$ be a closed $\overline{n}-$path in Σ. If c has a length $q > 1$, then $\mathcal{I}_{\pi,Sh(c)}(k,1) = \mathcal{I}_{\pi,c}(k,0)+\mathcal{I}_{\pi,c}(k,q)$ for all $k \in \{0,\dots,p\}$.*

Corollary 6.12 *Let $\pi = (y_k)_{k=0,\dots,p}$ be an $n-$path and $c = (x_i)_{i=0,\dots,q}$ be closed $\overline{n}-$path in Σ, then $\mathcal{I}_{\pi,c} = \mathcal{I}_{\pi,Sh(c)}$.*

Proof of Lemma 6.11 : Let $Sh(c) = (x'_0,\dots,x'_q)$ so that $x_{q-1} = x'_0$, $x_0 = x'_1$ and $x_1 = x'_2$.

We have $\mathcal{I}_{\pi,Sh(c)}(k,1) = \mathcal{I}^-_{\pi,Sh(c)}(k,1) + \mathcal{I}^+_{\pi,Sh(c)}(k,1)$. Since $x_0 = x'_1$ and $x_{q-1} = x'_0$ it follows that $\mathcal{I}^-_{\pi,Sh(c)}(k,1) = \mathcal{I}^-_{\pi,c}(k,q)$ for all $k \in \{0,\dots,p\}$. Furthermore, since $x_0 = x'_1$ and $x_1 = x'_2$ it follows that $\mathcal{I}^+_{\pi,Sh(c)}(k,1) = \mathcal{I}^+_{\pi,c}(k,0)$ for all $k \in \{0,\dots,p\}$.

Finally, $\mathcal{I}_{\pi,Sh(c)}(k,1) = \mathcal{I}^+_{\pi,c}(k,0)+\mathcal{I}^-_{\pi,c}(k,q)$ for all $k \in \{0,\dots,p\}$; but from Definition 6.3, $\mathcal{I}^+_{\pi,c}(k,0) = \mathcal{I}_{\pi,c}(k,0)$ and $\mathcal{I}^-_{\pi,c}(k,q) = \mathcal{I}_{\pi,c}(k,q)$. □

Then, in order to prove Proposition 6.2 we will need the two following lemmas which state the behavior of the contributions to the intersection number at the extremities of each path π and c of the Proposition.

Lemma 6.13 *Let $\pi = (y_k)_{k=0,\dots,p}$ be a closed $n-$path and $c = (x_i)_{i=0,\dots,q}$ be an $\overline{n}-$path in Σ. Then $\mathcal{I}_{\pi,c}(0,i) = -(\mathcal{I}_{c,\pi}(i,0) + \mathcal{I}_{c,\pi}(i,p))$ for all $i \in \{1,\dots,q-1\}$.*

Proof : From Lemma 6.9, $\mathcal{I}_{\pi,c}(0,i) = \mathcal{I}_{Sh(\pi),c}(1,i)$. From Lemma 6.8 and for all $i \in \{1,\dots,q-1\}$, $\mathcal{I}_{Sh(\pi),c}(1,i) = -\mathcal{I}_{c,Sh(\pi)}(i,1)$. Now, since π is closed and from Lemma 6.11, $-\mathcal{I}_{c,Sh(\pi)}(i,1) = -(\mathcal{I}_{c,\pi}(i,0) + \mathcal{I}_{c,\pi}(i,p))$. □

Lemma 6.14 *Let $\pi = (y_k)_{k=0,\dots,p}$ be closed $n-$path and $c = (x_i)_{i=0,\dots,q}$ be closed $\overline{n}-$path in Σ. Then $\mathcal{I}_{\pi,c}(0,0) + \mathcal{I}_{\pi,c}(0,q) = -(\mathcal{I}_{c,\pi}(0,0) + \mathcal{I}_{c,\pi}(0,p))$*

Proof : From Lemma 6.9, $\mathcal{I}_{\pi,c}(0,0) + \mathcal{I}_{\pi,c}(0,q) = \mathcal{I}_{Sh(\pi),c}(1,0) + \mathcal{I}_{Sh(\pi),c}(1,q)$ and from Lemma 6.11, $\mathcal{I}_{Sh(\pi),c}(1,0) + \mathcal{I}_{Sh(\pi),c}(1,q) = \mathcal{I}_{Sh(\pi),Sh(c)}(1,1)$. Then, following Lemma 6.8, $\mathcal{I}_{Sh(\pi),Sh(c)}(1,1) = -\mathcal{I}_{Sh(c),Sh(\pi)}(1,1)$. Again, Lemma 6.9 implies that $-\mathcal{I}_{Sh(c),Sh(\pi)}(1,1) = -\mathcal{I}_{c,Sh(\pi)}(0,1)$ whereas Lemma 6.11 implies that $-\mathcal{I}_{c,Sh(\pi)}(0,1) = -(\mathcal{I}_{c,\pi}(0,0)+\mathcal{I}_{c,\pi}(0,p))$. Finally, we have obtained that $\mathcal{I}_{\pi,c}(0,0) + \mathcal{I}_{\pi,c}(0,q) = -(\mathcal{I}_{c,\pi}(0,0) + \mathcal{I}_{c,\pi}(0,p))$. □

Proof of Proposition 6.2 : The sum of Definition 6.4 may be written as follows :

$$\begin{aligned}\mathcal{I}_{\pi,c} &= \mathcal{I}_{\pi,c}(0,0)+\mathcal{I}_{\pi,c}(0,q)+\sum_{i=1}^{q-1}\mathcal{I}_{\pi,c}(0,i)+ \qquad (6.3)\\ &\sum_{k=1}^{p-1}\left[\mathcal{I}_{\pi,c}(k,0)+\mathcal{I}_{\pi,c}(k,q)+\sum_{i=1}^{q-1}\mathcal{I}_{\pi,c}(k,i)\right]\end{aligned}$$

- If π is closed then Lemma 6.13 implies that $\mathcal{I}_{\pi,c}(0,i) = -\mathcal{I}_{c,\pi}(i,0) - \mathcal{I}_{c,\pi}(i,p)$ for $i \in \{1,\dots,q-1\}$. Then, $\sum_{i=1}^{q-1}\mathcal{I}_{\pi,c}(0,i) = -\sum_{i=1}^{q-1}[\mathcal{I}_{c,\pi}(i,0)+\mathcal{I}_{c,\pi}(i,p)]$. Furthermore, from Lemma 6.8, $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k)$ for all $k \in \{1,\dots,p-1\}$ and all $i \in \{1,\dots,q-1\}$. Thus, equation (6.3) becomes :

$$\begin{aligned}\mathcal{I}_{\pi,c} &= \mathcal{I}_{\pi,c}(0,0)+\mathcal{I}_{\pi,c}(0,q)+\sum_{i=1}^{q-1}-(\mathcal{I}_{c,\pi}(i,0)+\mathcal{I}_{c,\pi}(i,p))+ \qquad (6.4)\\ &\sum_{k=1}^{p-1}\left[\mathcal{I}_{\pi,c}(k,0)+\mathcal{I}_{\pi,c}(k,q)+\sum_{i=1}^{q-1}\mathcal{I}_{c,\pi}(i,k)\right]\end{aligned}$$

  – If c is closed, then Lemma 6.13 implies that $\mathcal{I}_{c,\pi}(0,k) = -(\mathcal{I}_{\pi,c}(k,0)+\mathcal{I}_{\pi,c}(k,q))$ for $k \in \{1,\dots,p-1\}$ so that equation (6.4) becomes :

$$\begin{aligned}\mathcal{I}_{\pi,c} &= \mathcal{I}_{\pi,c}(0,0)+\mathcal{I}_{\pi,c}(0,q)-\sum_{i=1}^{q-1}[\mathcal{I}_{c,\pi}(i,0)+\mathcal{I}_{c,\pi}(i,p)]+ \qquad (6.5)\\ &\sum_{k=1}^{p-1}\left[-\mathcal{I}_{c,\pi}(0,k)-\sum_{i=1}^{q-1}\mathcal{I}_{c,\pi}(i,k)\right]\end{aligned}$$

  If π and c are closed paths, it follows from Lemma 6.14 that $\mathcal{I}_{\pi,c}(0,0)+\mathcal{I}_{\pi,c}(0,q) = \mathcal{I}_{c,\pi}(0,0)+\mathcal{I}_{c,\pi}(0,p)$. Then,

$$\begin{aligned}\mathcal{I}_{\pi,c} &= -(\mathcal{I}_{c,\pi}(0,0)+\mathcal{I}_{c,\pi}(0,p))-\sum_{i=1}^{q-1}[\mathcal{I}_{c,\pi}(i,0)+\mathcal{I}_{c,\pi}(i,p)]+ \qquad (6.6)\\ &\sum_{k=1}^{p-1}\left[-\mathcal{I}_{c,\pi}(0,k)-\sum_{i=1}^{q-1}\mathcal{I}_{c,\pi}(i,k)\right]\\ \mathcal{I}_{\pi,c} &= -\sum_{i=0}^{q-1}[\mathcal{I}_{c,\pi}(i,0)+\mathcal{I}_{c,\pi}(i,p)]-\sum_{k=1}^{p-1}\sum_{i=0}^{q-1}\mathcal{I}_{c,\pi}(i,k) \qquad (6.7)\\ \mathcal{I}_{\pi,c} &= -\sum_{i=0}^{q-1}\mathcal{I}_{c,\pi}(i,0)-\sum_{i=0}^{q-1}\mathcal{I}_{c,\pi}(i,p)-\sum_{k=1}^{p-1}\sum_{i=0}^{q-1}\mathcal{I}_{c,\pi}(i,k) \qquad (6.8)\\ \mathcal{I}_{\pi,c} &= -\sum_{k=0}^{p}\sum_{i=0}^{q-1}\mathcal{I}_{c,\pi}(i,k) = -\mathcal{I}_{c,\pi}\end{aligned}$$

– If c is not closed and since $\mathcal{P}(c,\pi)$ holds, then $\mathcal{I}_{c,\pi}(0,k) = \mathcal{I}_{\pi,c}(k,0) = \mathcal{I}_{\pi,c}(k,q) = 0$ for all $k \in \{0,\dots,p\}$. Then, equation (6.4) becomes :

$$\begin{aligned}
\mathcal{I}_{\pi,c} &= \sum_{i=1}^{q-1} -(\mathcal{I}_{c,\pi}(i,0) + \mathcal{I}_{c,\pi}(i,p)) - \sum_{k=1}^{p-1}\sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,k) \\
\mathcal{I}_{\pi,c} &= -\sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,0) - \sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,p)) - \sum_{k=1}^{p-1}\sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,k) \\
\mathcal{I}_{\pi,c} &= -\sum_{k=0}^{p}\sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,k) = -\sum_{i=1}^{q-1}\sum_{k=0}^{p} \mathcal{I}_{c,\pi}(i,k)
\end{aligned}$$

Since $\mathcal{I}_{c,\pi}(0,k) = 0$ for all $k \in \{0,\dots,p\}$,

$$\mathcal{I}_{\pi,c} = -\sum_{i=0}^{q-1}\sum_{k=0}^{p} \mathcal{I}_{c,\pi}(i,k) = -\mathcal{I}_{c,\pi}$$

• If π is not closed and since $\mathcal{P}(\pi,c)$ holds, then $\mathcal{I}_{\pi,c}(0,i) = \mathcal{I}_{\pi,c}(p,i) = \mathcal{I}_{c,\pi}(i,0) = \mathcal{I}_{c,\pi}(i,p) = 0$ for $i \in \{0,\dots,q\}$ so that $\sum_{i=0}^{q} \mathcal{I}_{\pi,c}(0,i) = 0$. Then, equation (6.3) becomes :

$$\mathcal{I}_{\pi,c} = \sum_{k=1}^{p-1}\left[\mathcal{I}_{\pi,c}(k,0) + \mathcal{I}_{\pi,c}(k,q) + \sum_{i=1}^{q-1} \mathcal{I}_{\pi,c}(k,i)\right]$$

From Lemma 6.8, $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k)$ for all $k \in \{1,\dots,p-1\}$ and all $i \in \{1,\dots,q-1\}$.

$$\mathcal{I}_{\pi,c} = \sum_{k=1}^{p-1}\left[\mathcal{I}_{\pi,c}(k,0) + \mathcal{I}_{\pi,c}(k,q) - \sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,k)\right] \tag{6.10}$$

– If c is closed, then Lemma 6.13 implies that $\mathcal{I}_{\pi,c}(k,0) + \mathcal{I}_{\pi,c}(k,q) = -\mathcal{I}_{c,\pi}(0,k)$ for all $k \in \{1,\dots,p-1\}$ and equation (6.10) becomes :

$$\mathcal{I}_{\pi,c} = \sum_{k=1}^{p-1}\sum_{i=0}^{q-1} -\mathcal{I}_{c,\pi}(i,k) = \sum_{i=0}^{q-1}\sum_{k=1}^{p-1} -\mathcal{I}_{c,\pi}(i,k)$$

Furthermore, $\mathcal{I}_{c,\pi}(i,0) = \mathcal{I}_{c,\pi}(i,p) = 0$ for all $i \in \{0,\dots,q\}$ since π is not closed and $\mathcal{P}(\pi,c)$ holds, so that :

$$\mathcal{I}_{\pi,c} = \sum_{i=0}^{q-1}\sum_{k=0}^{p} -\mathcal{I}_{c,\pi}(i,k) = -\mathcal{I}_{c,\pi}$$

– If c is not closed then $\mathcal{I}_{\pi,c}(k,0) = \mathcal{I}_{\pi,c}(k,q) = 0$ for all $k \in \{0,\dots,p\}$ since $\mathcal{P}(c,\pi)$ holds, then equation (6.10) becomes :

$$\mathcal{I}_{\pi,c} = -\sum_{k=1}^{p-1}\sum_{i=1}^{q-1} \mathcal{I}_{c,\pi}(i,k)$$

From Lemma 6.8, $\mathcal{I}_{\pi,c}(k,i) = -\mathcal{I}_{c,\pi}(i,k)$ for all $k \in \{1,\ldots,p-1\}$ and all $i \in \{1,\ldots,q-1\}$. Then,

$$\mathcal{I}_{\pi,c} = -\sum_{i=1}^{q-1}\sum_{k=1}^{p-1}\mathcal{I}_{c,\pi}(i,k) = -\mathcal{I}_{c,\pi}$$

□

6.3.3 An additive property

The following proposition will be useful in further proofs.

Proposition 6.15 *Let $\pi = (y_k)_{k=0,\ldots,p}$ be an $n-$path on Σ; let $c = (x_i)_{i=0,\ldots,q}$ and $c' = (x'_i)_{i=0,\ldots,q'}$ be two $\overline{n}-$paths on Σ such that $x_q = x'_0$. If $\mathcal{P}(\pi,c)$ and $\mathcal{P}(\pi,c')$ hold, then $\mathcal{I}_{\pi,c.c'} = \mathcal{I}_{\pi,c} + \mathcal{I}_{\pi,c'}$.*

Proof of Proposition 6.15 : Let us compute $\mathcal{I}_{\pi,c.c'}$ with $c.c' = (z_0,\ldots,z_{q+q'})$.

It is sufficient to prove that for $k \in \{1,\ldots,p-1\}$ ($k \in \{0,\ldots,p\}$ if π is closed) :

$$\sum_{i=0}^{q+q'}\mathcal{I}_{\pi,c.c'}(k,i) = \sum_{i=0}^{q}\mathcal{I}_{\pi,c}(k,i) + \sum_{i=0}^{q'}\mathcal{I}_{\pi,c'}(k,i) \tag{6.11}$$

We simply write that for $k \in \{0,\ldots,p-1\}$ ($k \in \{0,\ldots,p\}$ if π is closed) :

$$\begin{aligned}\sum_{i=0}^{q+q'}\mathcal{I}_{\pi,c.c'}(k,i) &= \mathcal{I}_{\pi,c.c'}(k,0) + \left[\sum_{i=1}^{q-1}\mathcal{I}_{\pi,c.c'}(k,i)\right] + \mathcal{I}_{\pi,c.c'}(k,q) \\ &+ \left[\sum_{i=q+1}^{q+q'-1}\mathcal{I}_{\pi,c.c'}(k,i)\right] + \mathcal{I}_{\pi,c.c'}(k,q+q')\end{aligned} \tag{6.12}$$

Now, for such k we observe that $\mathcal{I}_{\pi,c}(k,0) = \mathcal{I}^+_{\pi,c}(k,0)$ from Definition 6.3. Since $x_0 = z_0$ and $x_1 = z_1$ we obtain that $\mathcal{I}^+_{\pi,c}(k,0) = \mathcal{I}^+_{\pi,c.c'}(k,0)$ which is also equal to $\mathcal{I}_{\pi,c.c'}(k,0)$ following Definition 6.3. By the same way, we prove that $\mathcal{I}_{\pi,c.c'}(k,q+q') = \mathcal{I}_{\pi,c'}(k,q')$.

For $i \in \{1,\ldots,q-1\}$, we have $\mathcal{I}_{\pi,c.c'}(k,i) = \mathcal{I}_{\pi,c}(k,i)$ since $x_i = z_i$, $x_{i-1} = z_{i-1}$ and $x_{i+1} = z_{i+1}$. Similarly, for $i \in \{q+1,\ldots,q+q'-1\}$, we have $\mathcal{I}_{\pi,c.c'}(k,i) = \mathcal{I}_{\pi,c'}(k,i-q)$ since $x_i = z_{i-q}$, $x_{i-1} = z_{(i-q)-1}$ and $x_{i+1} = z_{(i-q)+1}$.

Furthermore, we have $\mathcal{I}_{\pi,c.c'}(k,q) = \mathcal{I}^-_{\pi,c.c'}(k,q) + \mathcal{I}^+_{\pi,c.c'}(k,q)$. Then, we observe that $\mathcal{I}_{\pi,c}(k,q) = \mathcal{I}^-_{\pi,c}(k,q)$ and $\mathcal{I}_{\pi,c'}(k,0) = \mathcal{I}^+_{\pi,c'}(k,0)$. But, $\mathcal{I}^-_{\pi,c}(k,q) = \mathcal{I}^-_{\pi,c.c'}(k,q)$ since $x_q = z_q$ and $x_{q-1} = z_{q-1}$. Similarly, $\mathcal{I}^+_{\pi,c'}(k,0) = \mathcal{I}^+_{\pi,c.c'}(k,q)$ since $x'_0 = z_q$ and $x'_1 = z_{q+1}$. Finally, $\mathcal{I}_{\pi,c.c'}(k,q) = \mathcal{I}_{\pi,c}(k,q) + \mathcal{I}_{\pi,c'}(k,0)$.

By replacing the corresponding terms in equation (6.12) we obtain that :

$$\begin{aligned}\sum_{i=0}^{q+q'} \mathcal{I}_{\pi,c.c'}(k,i) &= \mathcal{I}_{\pi,c'}(k,0) + \left[\sum_{i=1}^{q-1} \mathcal{I}_{\pi,c}(k,i)\right] + \mathcal{I}_{\pi,c}(k,q) \\ &+ \mathcal{I}_{\pi,c'}(k,0) + \left[\sum_{i=1}^{q'-1} \mathcal{I}_{\pi,c'}(k,i)\right] + \mathcal{I}_{\pi,c'}(k,q')\end{aligned}$$

or,

$$\sum_{i=0}^{q+q'} \mathcal{I}_{\pi,c.c'}(k,i) = \sum_{i=0}^{q} \mathcal{I}_{\pi,c}(k,i) + \sum_{i=0}^{q'} \mathcal{I}_{\pi,c'}(k,i)$$

Finally, $\mathcal{I}_{\pi,c.c'} = \mathcal{I}_{\pi,c} + \mathcal{I}_{\pi,c'}$. $\square$

Corollary 6.16 *Let $\pi = (y_k)_{k=0,\ldots,p}$ and $\pi' = (y'_k)_{k=0,\ldots,p'}$ be two $n-$paths on a digital surface Σ such that $y_p = y'_0$; let $c = (x_i)_{i=0,\ldots,q}$ be an $\overline{n}-$path on Σ . If $\mathcal{P}(\pi,c)$, $\mathcal{P}(\pi',c)$, $\mathcal{P}(c,\pi)$ and $\mathcal{P}(c,\pi')$ hold then $\mathcal{I}_{\pi.\pi',c} = \mathcal{I}_{\pi,c} + \mathcal{I}_{\pi',c}$.*

Proof : Since $\mathcal{P}(c,\pi)$ and $\mathcal{P}(c,\pi)$ hold it is immediate that $\mathcal{P}(c,\pi.\pi')$ holds. Then, from Proposition 6.2, we have $\mathcal{I}_{\pi.\pi',c} = \mathcal{I}_{c,\pi.\pi'}$. Now, from Proposition 6.15 we obtain that $\mathcal{I}_{c,\pi.\pi'} = \mathcal{I}_{c,\pi} + \mathcal{I}_{c,\pi'}$. But, under the hypothesis of this corollary and again from Proposition 6.2 we have $\mathcal{I}_{c,\pi} = \mathcal{I}_{\pi,c}$ and $\mathcal{I}_{c,\pi'} = \mathcal{I}_{\pi',c}$. $\square$

6.4 Proof of the main Theorems

The proof of Theorem 9 will be slightly different for the case when $(n,\overline{n}) = (e,v)$ and $(n,\overline{n}) = (v,e)$. However, in both cases, we will first define a relation of deformation between paths (the which is in fact equivalent to the homotopy relation as stated respectively by Proposition 6.24 and Proposition 6.17 respectively for $n = v$ and $n = e$.

For $n = v$, this new deformation is based on the insertion of triplets of surfels, or the insertion of back and forth in the paths. Then, Proposition 6.18 will state that a triplet of surfel always have an intersection number equal to zero with any $e-$path (as soon as this $e-$path is closed, otherwise the triplet must not meet an extremity of the $e-$path).

For $n = e$, this new deformation is based on the insertion of $e-$loops of surfels (Definition 6.10), or the insertion of back and forth in the paths. Then, Proposition 6.25 will state in a similar way to Proposition 6.18 that an $e-$loop always have an intersection number equal to zero with any $v-$path (a soon as this $v-$path is closed, otherwise the $e-$loop must not meet an extremity of the $v-$path).

Finally, using Proposition 6.15, a straightforward proof of Theorem 9 for $n = e$ and $n = v$ will be given.

Remark 6.6 *Note that, without loss of generality, we suppose in this section that any path mentioned (except closed ones) has the following property : two consecutive surfels in the path are distinct.*

6.4.1 Another definition for the homotopy of $v-$paths

First, we introduce the notion of an elementary $\mathcal{T}-$deformation and the definition of the $\mathcal{T}-$deformation relation follows immediately.

Definition 6.7 (back and forth) *A simple closed $n-$path $\pi = (x_0, x_1, x_0)$ in Σ is called a* back and forth *in Σ.*

Definition 6.8 (triplet) *A simple closed $v-$path $\pi = (x_0, x_1, x_2, x_0)$ included in a loop of Σ is called a* triplet *in Σ.*

Definition 6.9 *Let $X \subset \Sigma$, $c = (x_i)_{i=0,\dots,q}$ and $c' = (x'_i)_{i=0,\dots,q'}$ be two $v-$paths in X. The path c is said to be an* elementary $\mathcal{T}-$deformation *of c' in X (and we denote $c \sim_{\mathcal{T}} c'$) if $c = \pi_1.(s).\pi_2$ and $c' = \pi_1.\gamma.\pi_2$; or if $c = \pi_1.\gamma.\pi_2$ and $c' = \pi_1.(s).\pi_2$. Where γ is a back and forth from s to s in X, or γ is a triplet from s to s in X. We define the $\mathcal{T}-$deformation relation as the transitive closure of the elementary $\mathcal{T}-$deformation relation.*

In other words, the relation of elementary $\mathcal{T}-$deformation links together two $v-$paths which are almost the same except that one is obtained by insertion in the other of a triplet of surfels which belongs to the same loop , or by insertion in the other of a *back and forth.* Now, we can state the following proposition :

Proposition 6.17 *Let $X \subset \Sigma$. Two $v-$paths c and c' are $v-$homotopic in X if and only if they are the same up to a $\mathcal{T}-$deformation.*

Proof : First, an elementary $\mathcal{T}-$deformation is a particular case of an elementary $v-$deformation where the $v-$paths γ and γ' of Definition 2.10 are closed paths, one of which is reduced to a single surfel and the other one is a triplet or a closed path with a length of 2 which are both included in a loop, i.e. an elementary deformation cell. It immediately follows that if two $v-$paths are the same up to a $\mathcal{T}-$deformation then they are $v-$homotopic.

Now, it is sufficient to prove that, if two $v-$paths are the same up to an elementary $v-$deformation, then they are the same up to a $\mathcal{T}-$deformation. Let c and c' be two $v-$paths which are the same up to an elementary $v-$deformation, i.e. $c = \pi_1.\gamma.\pi_2$ and $c' = \pi_1.\gamma'.\pi_2$ where γ and γ' are two paths with the same extremities and included in a common loop.

We first prove that any $v-$path $\alpha = (a_0, \ldots, a_h)$ with a length l greater than one and included in a loop is a $\mathcal{T}-$deformation of the path (a_0, a_h). We proceed by an induction on the length l. Let α_k be a $v-$path included in a loop $\mathcal{L}$ with a length l_k. We distinguish two cases :

- Either $\alpha_k = (a_0, a_1)$, or
- α_k is a path with a length $l_k \geq 2$. Then $\alpha_k = \omega.(a, b, c)$ where $\{a, b, c\} \subset \mathcal{L}$ and ω may be reduced to (a) if $l = 2$. Then the path α_k is an elementary $\mathcal{T}-$deformation of the path $\alpha' = \omega.(a, b, c).(c, b, a, c).(c)$. Now, $\alpha' = \omega.(a, b).(b, c, b).(b, a, c)$ is an elementary $\mathcal{T}-$deformation of the path $\alpha'' = \omega.(a, b, a, c)$ which is itself an elementary $\mathcal{T}-$deformation of the path $\alpha_{k+1} = \omega.(a, c)$. Finally, α_k is an $\mathcal{T}-$deformation of the path $\alpha_{k+1} = \omega.(a, c)$, which has a length $l_{k+1} = l_k - 1$.

Finally, either the path α_k has a length of one or it is shown that α_k is a $\mathcal{T}-$deformation of a path α_{k+1} in X with a lower length then α_k. By induction with $\alpha_0 = \alpha$, their must exist $k' > 0$ such that $\alpha_{k'}$ has a length of one (i.e $\alpha_{k'} = (a_0, a_h)$) and which is a $\mathcal{T}-$deformation of α.

Then, we have just proved that both paths γ and γ' are equivalent up to a $\mathcal{T}-$deformation in X to the path reduced to their extremities. It is then immediate that those two paths are themselves equivalent up to a $\mathcal{T}-$deformation in X. $\square$

Intersection number of triplets

In this subsection, we will prove the following Proposition which states that a triplet c (Definition 6.8) has an intersection number equal to zero with any $e-$path π as soon as $\mathcal{P}(\pi, c)$ holds.

Proposition 6.18 *Let c be a triplet in Σ and let $\pi = (y_k)_{k=0,\ldots,p}$ be an $e-$path on Σ such* *that $\mathcal{P}(\pi, c)$ holds. Then, $\mathcal{I}_{\pi,c} = 0$.*

In order to prove Proposition 6.18, we first state the two following lemmas.

Lemma 6.19 *Let $c = (x_0, x_1, x_2, x_0)$ be a triplet in Σ. Then, depending on the order of the parameterization of c, one of the two following properties is satisfied :*

- $\forall i \in \{0, 1, 2\}$, $Left_c(i) \cap N_e(x_i) = \emptyset$.
- $\forall i \in \{0, 1, 2\}$, $Right_c(i) \cap N_e(x_i) = \emptyset$.

Proof : This lemma comes from local considerations. Indeed, since for all $i \in \{0, 1, 2\}$ the three surfels x_{i-1}, x_i and x_{i+1} are included in a common loop, then, depending on the order of the parameterization, exactly one of the intervals between x_{i-1} and x_{i+1} of the canonical parameterization of $N_v(x_i)$ cannot contain a surfel $e-$adjacent to x_i. And it is readily seen that this interval coincides either with $Left_c(i)$ for all $i \in \{0, 1, 2\}$ or with $Right_c(i)$ for all $i \in \{0, 1, 2\}$. □

Lemma 6.20 *Let $c = (x_0, x_1, x_2, x_0)$ be a triplet in Σ. Then, one of the two following properties is satisfied :*

- $\mathcal{I}_{c,\pi} = -0.5$ *for all $e-$path π with a length 1 which enters c and* $\mathcal{I}_{c,\pi} = +0.5$ *for all $e-$path π with a length 1 which exits c.*
- $\mathcal{I}_{c,\pi} = +0.5$ *for all $e-$path π with a length 1 which enters c and* $\mathcal{I}_{c,\pi} = -0.5$ *for all $e-$path π with a length 1 which exits c.*

Proof : This lemma is a straightforward consequence of Lemma 6.19 and Definition 6.3. □

Proof of Proposition 6.18 : Let $\pi = (y_k)_{k=0,\dots,p}$ and $\pi_h = (y_h, y_{h+1})$ for $h \in \{0, \dots, p-1\}$ so that $\pi = \pi_0.\pi_1.\dots.\pi_{p-1}$. Since c is closed, then property $\mathcal{P}(c, \pi')$ holds and since $\mathcal{P}(\pi, c)$ holds too, from Proposition 6.2 we have $\mathcal{I}_{\pi,c} = -\mathcal{I}_{c,\pi}$.

Furthermore, since c is closed, the property $\mathcal{P}(c, \pi')$ holds for any $e-$path π' in Σ. Then, Proposition 6.15 implies :

$$\mathcal{I}_{c,\pi} = \mathcal{I}_{c,\pi_0} + \mathcal{I}_{c,\pi_1} + \ldots + \mathcal{I}_{c,\pi_{q-1}}$$

First, it is immediate that $\mathcal{I}_{c,\pi_h} = 0$ for any $h \in \{0, \dots, p-1\}$ such that π_h does not enter neither exits c. Indeed, $\mathcal{I}_{c,\pi_h}$ is obviously equal to 0 if $c^* \cap \pi_h^* = \emptyset$; and since c has a length of 3 it is also immediate (from the Definition 6.3) that $\mathcal{I}_{c,\pi_h} = 0$ when $\pi_h^* \subset c^*$. Furthermore, since π is either closed or c^* meets neither y_0 nor y_p (property $\mathcal{P}(\pi, c)$), it is immediate that the number of π_h's which enter c is equal to the number of π_h's which exit c. Then, from Lemma 6.20, it follows that $\mathcal{I}_{c,\pi} = \sum_{h=0}^{p-1} \mathcal{I}_{c,\pi_h} = 0$. Finally, $\mathcal{I}_{\pi,c} = -\mathcal{I}_{c,\pi} = 0$.
□

Remark 6.7 *The intersection number $\mathcal{I}_{\pi,c}$ of an $e-$path π with a triplet c is either equal to zero or not defined. Indeed, if $\mathcal{P}(\pi, c)$ is not satisfied, then $\mathcal{I}_{\pi,c}$ is not defined (see Figure 6.13).*

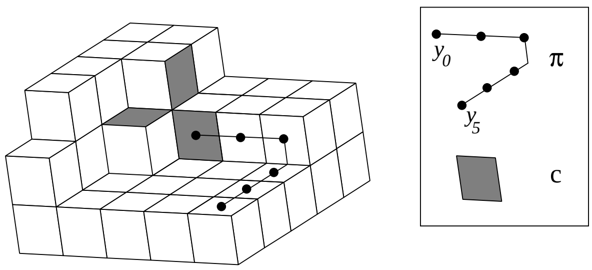


Figure 6.13: $\mathcal{I}_{\pi,c}$ is not defined since $y_0 \in c^*$, whereas $\mathcal{I}_{c,\pi} = \pm 0.5$.

Now, we can achieve the proof of Theorem 9 for $n = e$ using Proposition 6.17, Proposition 6.15 and Proposition 6.18.

6.4.2 Proof of Theorem 9 when $(n, \overline{n}) = (e, v)$

Here, we achieve the proof of the main theorem in the case of a $v-$homotopic deformation of the $v-$path c.

Proof of Theorem 9 for $(n, \overline{n}) = (e, v)$: From Proposition 6.17 it is sufficient to prove Theorem 9 in the case when c' is an elementary $\mathcal{T}-$deformation of c in X. Following Definition 6.9, we may suppose that $c = c_1.(s).c_2$ and $c' = c_1.\gamma.c_2$. Where γ is a back and forth or a triplet from s to s in X. Since $\mathcal{P}(\pi, c)$ holds, it is straightforward that $\mathcal{P}(\pi, c_1)$, $\mathcal{P}(\pi, \gamma)$ and $\mathcal{P}(\pi, c_2)$ hold too. Then, from Proposition 6.15, we have $\mathcal{I}_{\pi,c} = \mathcal{I}_{\pi,c_1} + \mathcal{I}_{\pi,\gamma} + \mathcal{I}_{\pi,c_2}$.

If γ is a back and forth in X, i.e $\gamma = (\gamma^0, \gamma^1, \gamma^3)$ where $\gamma^3 = \gamma^0$, then it is immediate from Definition 6.3 that $\mathcal{I}^{\pi}_{\pi,\gamma}(0) = 0$ and $\mathcal{I}^{\pi}_{\pi,\gamma}(1) = 0$ so that $\mathcal{I}_{\pi,\gamma} = 0$. On the other hand, if γ is a triplet, then $\mathcal{I}_{\pi,\gamma} = 0$ from Proposition 6.18.

In both cases, it remains that $\mathcal{I}_{\pi,c} = \mathcal{I}_{\pi,c_1} + \mathcal{I}_{\pi,c_2} = \mathcal{I}_{\pi,c_1.c_2} = \mathcal{I}_{\pi,c'}$. $\square$

6.4.3 Another definition of homotopy for $e-$paths

Definition 6.10 ($e-$loop) *A parameterization (see Definition 4.7) of a loop containing a surfel x of Σ and which starts at the surfel x is called an* $e-$loop from x to x in Σ *(see Figure 6.14).*

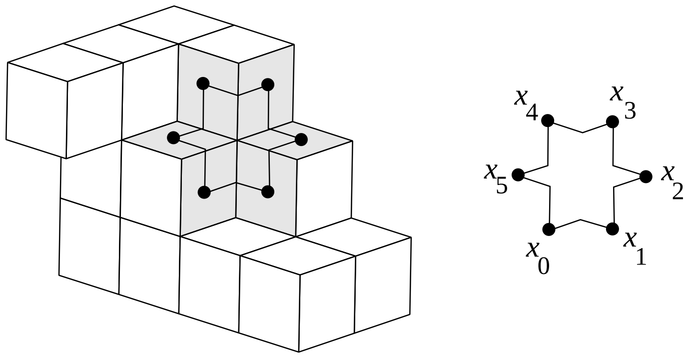


Figure 6.14: An $e-$loop $c = (x_0, x_1, x_2, x_3, x_4, x_5, x_0)$ in a digital surface Σ.

First, we introduce the notion of an elementary $\mathcal{E}-$deformation and the definition of the $\mathcal{E}-$deformation relation follows immediately.

Definition 6.11 *Let $X \subset \Sigma$, c and c' be two $e-$paths in X. The path c is said to be an* elementary $\mathcal{E}-$deformation *of c' in X (and we denote $c \sim_{\mathcal{E}} c'$) if $c = c_1.(s).c_2$ and $c' = c_1.\gamma.c_2$; or if $c = c_1.\gamma.c_2$ and $c' = c_1.(s).c_2$. Where γ is an $e-$loop or a back and forth from s to s in X. In this case, we also say that c and c' are the same up to an elementary $\mathcal{E}-$deformation. We define the $\mathcal{E}-$deformation relation (denoted by $\simeq_{\mathcal{E}}$) as the transitive closure of the elementary $\mathcal{E}-$deformation relation.*

In other words, the relation of elementary $\mathcal{E}-$deformation links together two $e-$paths which are almost the same except that one is obtained by insertion in the other of a simple closed $e-$path included in a loop. Now, we can state the following proposition :

Lemma 6.21 *Let c be an $e-$path in Σ. Then, either c is simple or $c = c_1.\beta.c_2$ where β is a simple closed path with a length greater then 1.*

Proof : Let $c = (x_0, \ldots, x_q)$. Then, if c is not simple, let $h \in \{0, \ldots, q\}$ and $l \in \{0, \ldots, q\}$ be such that $x_h = x_l$ and $l > h$; and suppose that l is minimal for these properties. Thus, $c = (x_0, \ldots, x_h).(x_h, \ldots, x_l).(x_l, \ldots, x_q)$ and $(x_h, \ldots, x_l)$ is simple. $\square$

Lemma 6.22 *Let $c = (x_i)_{i=0,\ldots,q}$ be an $e-$path in X included in a loop $\mathcal{L}$ of Σ. Then c is $\mathcal{E}-$deformation of a simple path from x_0 to x_q in X.*

Proof : We proceed by an induction on the length of a path α^k for $k \geq 0$ with $\alpha^0 = c$. Let α^k be an $e-$path in X with a length l_k and which is included in $\mathcal{L}$. From Lemma 6.21, α^k is simple or there exists a simple closed path β^k with a length greater than 1 such that $\alpha^k = \alpha_1^k.\beta^k.\alpha_2^k$. Since β_k is obviously included in $\mathcal{L}$, then β^k is an $e-$loop or a back and forth in X so that α^k is an elementary $\mathcal{E}-$deformation of the path $\alpha^{k+1} = \alpha_1^k.\alpha_2^k$. Furthermore, the path α^{k+1} has a length $l_{k+1} < l_k$ since β^k has a length greater than 1. Since the length l_k is necessary greater than or equal to 0, it follows that there exists an integer $l \geq 0$ such that α^l is simple. Furthermore, for $i = 0, \ldots, l-1$, the path α^{i+1} is an elementary $\mathcal{E}-$deformation of α^i so that α^l is an $\mathcal{E}-$deformation of $\alpha^0 = \pi$. $\square$

Lemma 6.23 *Let $c = (x_i)_{i=0,\ldots,q}$ be an $e-$path in X. The path $c.c^{-1}$ is an $\mathcal{E}-$deformation of the trivial path (x_0, x_0).*

Proof : The proof of this lemma is similar to the proof of Lemma 2.4.

$\square$

Proposition 6.24 *Let $X \subset \Sigma$. Two $e-$paths c and c' are $e-$homotopic in X if and only if they are the same up to an $\mathcal{E}-$deformation.*

Proof : First, an elementary $\mathcal{E}-$deformation is a particular case of an elementary $e-$deformation where the $e-$paths γ and γ' of Definition 2.10 are closed paths, one of which is reduced to a single surfel and the other one is simple closed $e-$path included in a loop, i.e. an elementary deformation cell. It immediately follows that if two $e-$paths are the same up to a $\mathcal{E}-$deformation then they are $e-$homotopic.

Now, it is sufficient to prove that, if two $e-$paths are the same up to an elementary $e-$deformation in X, then they are the same up to a $\mathcal{E}-$deformation in X. Let c and c' be two $e-$paths which are the same up to an elementary $e-$deformation, i.e. $c = c_1.\gamma.c_2$

and $c' = c_1.\gamma'.c_2$ where γ and γ' are two paths in X with the same extremities and included in a common loop $\mathcal{L}$.

From Lemma 6.22, the path γ [resp. γ'] is an $\mathcal{E}-$deformation of a simple path β [resp. β'] included in $\mathcal{L}$. Then, $c \simeq_{\mathcal{E}} c_1.\beta.c_2$ and $c' \simeq c_1.\beta'.c_2$ where β and β' are simple paths.

If $\beta = \beta'$, then it is immediate that $c \simeq_{\mathcal{E}} c'$.

Now, if β and β' are not the same but are both closed, then β and β' are simple closed $e-$paths in $\mathcal{L}$ so that $c_1.\beta.c_2 \sim_{\mathcal{E}} c_1.c_2$ and $c_1.\beta'.c_2 \sim_{\mathcal{E}} c_1.c_2$ (Lemma 6.22). Then, $c \simeq_{\mathcal{E}} (c_1.c_2) \simeq_{\mathcal{E}} c'$.

If β and β' are not the same and also not closed, let a and b be the two extremities of β which are distint. Now, from the very definition of a loop, there exists exactly two dictinct simple $e-$paths in a loop between two distinct surfels of this loop (see Figure 6.14). Since $\beta \neq \beta'$ then $\beta^* \cap \beta'^* = \{a, b\}$. It follows that the path $\beta^{-1}.\beta'$ is a simple closed path from b to b in $\mathcal{L}$. So $c_1.\beta \sim_{\mathcal{E}} c_1.\beta.\beta^{-1}.\beta'.c_2$. But from Lemma 6.23, $c_1.\beta.\beta^{-1}.\beta'.c_2 \simeq_{\mathcal{E}} c_1.\beta'.c_2$ so that $c_1.\beta.c_2 \simeq_{\mathcal{E}} c_1.\beta'.c_2$. Finally, $c \simeq_{\mathcal{E}} c'$. □

6.4.4 Intersection number of $e-$loops

Proposition 6.25 *Let c be an $e-$loop in Σ, then $\mathcal{I}_{c,\pi} = 0$ for any $v-$path π on Σ such that $\mathcal{P}(\pi, c)$ holds.*

Lemma 6.26 *Let $c = (x_i)_{i=0,\dots,q}$ be an $e-$loop in Σ. Then, depending on the order of the parameterization of c, one of the two following properties are satisfied :*

- $\forall i \in \{0, \dots, q\}$, $Left_c(i) \cap c^* = \emptyset$ *and* $Right_c(i) \subset c^*$.
- $\forall i \in \{0, \dots, q\}$, $Right_c(i) \cap c^* = \emptyset$ *and* $Left_c(i) \subset c^*$.

Proof : This lemma comes from local considerations following the very definition of the $e-$loops, the canonical parameterization of the $v-$neigbhborhood of a surfel, and the local left and right sets. □

Lemma 6.27 *Let $c = (x_i)_{i=0,\dots,q}$ be an $e-$loop in Σ. Then, depending on the order of the parameterization of c, one of the two following properties are satisfied :*

- $\mathcal{I}_{c,\pi} = -0.5$ *for all $v-$path π with a length 1 which enters c and* $\mathcal{I}_{c,\pi} = +0.5$ *for all $v-$path π with a length 1 which exits c.*

- $\mathcal{I}_{c,\pi} = +0.5$ *for all* $v-$*path* π *with a length 1 which enters* c *and* $\mathcal{I}_{c,\pi} = -0.5$ *for all* $v-$*path* π *with a length 1 which exits* c.

Proof : This lemma is a straightforward consequence of Lemma 6.26. $\square$

Proof of Proposition 6.25 : Let $\pi = (y_k)_{k=0,\dots,p}$ and $\pi_h = (y_h, y_{h+1})$ for $h \in \{0, \dots, p-1\}$ so that $\pi = \pi_0.\pi_1.\dots.\pi_{p-1}$. Since c is closed, the property $\mathcal{P}(c, \pi')$ holds for any $v-$path π' in Σ, then Proposition 6.15 implies that :

$$\mathcal{I}_{c,\pi} = \mathcal{I}_{c,\pi_0} + \mathcal{I}_{c,\pi_1} + \dots + \mathcal{I}_{c,\pi_{p-1}}$$

First, it is immediate that $\mathcal{I}_{c,\pi_h} = 0$ for any $h \in \{0, \dots, p-1\}$ such that π_h does not enter neither exits c. Indeed, $\mathcal{I}_{c,\pi_h}$ is obviously equal to 0 when when $\pi_h^* \cap c^* = \emptyset$; and in the case when $\pi_h^* \subset c^*$ then from Lemma 6.27 we also obtain that $\mathcal{I}_{c,\pi_h} = 0$. Furthermore, since π is either closed or c^* does meet neither y_0 nor y_p (property $\mathcal{P}(\pi, c)$), it is immediate that the number of π_h's which enter c is equal to the number of π_h's which exit c. Then, from Lemma 6.27, it follows that $\mathcal{I}_{c,\pi} = \sum_{h=0}^{p-1} \mathcal{I}_{c,\pi_h} = 0$. $\square$

Remark 6.8 *The intersection number* $\mathcal{I}_{c,\pi}$ *of an* $e-$*loop* c *with a* $v-$*path* π *may not be equal to zero if* $\mathcal{P}(\pi, c)$ *is not satisfied, as depicted in Figure 6.15.*

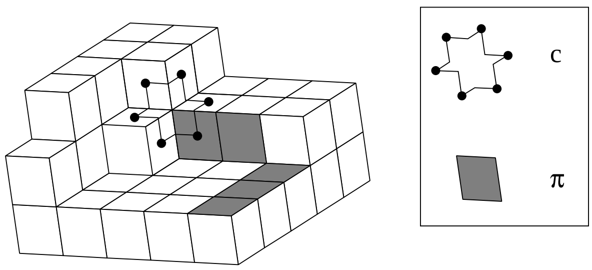


Figure 6.15: An $e-$loop c and a $v-$path π such that $\mathcal{I}_{c,\pi} = \pm 0.5$.

6.4.5 Proof of Theorem 9 when $(n, \overline{n}) = (v, e)$

Proof of Theorem 9 for $(n, \overline{n}) = (v, e)$ **:** Following Proposition 6.24, it is sufficient to prove that $\mathcal{I}_{\pi,c} = \mathcal{I}_{\pi,c'}$ when c and c' are two $e-$paths which are the same up to an elementary $\mathcal{E}-$deformation in X (in $X \setminus \{y_0, y_p\}$ if π is not closed). Then, let $c = c_1.\gamma.c_2$ and $c' = c_1.(s).c_2 = c_1.c_2$ where γ is an $e-$loop or a back and forth from s to s in X (in $X \setminus \{y_0, y_p\}$ if π is not closed).

Since $\mathcal{P}(\pi, c')$ holds, then is is immediate that $\mathcal{P}(\pi, c_1)$, $\mathcal{P}(\pi, \gamma)$ and $\mathcal{P}(\pi, c_2)$ hold too. Then, following Property 6.15, $\mathcal{I}_{\pi,c_1.\gamma.c_2} = \mathcal{I}_{\pi,c_1} + \mathcal{I}_{\pi,\gamma} + \mathcal{I}_{\pi,c_2}$.

If $\gamma = (y_0, y_1.y_2)$ where $y_2 = y_0$ is a back and forth in X (in $X \setminus \{y_0, y_p\}$ if π is not closed), then it is immediate from Definition 6.3 and Definition 6.4 that $\mathcal{I}_{\pi,\gamma} = 0$.

If γ is an $e-$loop, then $\mathcal{P}(\gamma, \pi)$ holds together with $\mathcal{P}(\pi, \gamma)$. Then, from Proposition 6.2, we have $\mathcal{I}_{\pi,\gamma} = -\mathcal{I}_{\gamma,\pi}$ and from Proposition 6.25 $\mathcal{I}_{\gamma,\pi} = 0$. Finally, $\mathcal{I}_{\pi,c'} = X_{\pi,c_1} + \mathcal{I}_{\pi,c_2}$ and from Proposition 6.15 it follows that $\mathcal{I}_{\pi,c'} = \mathcal{I}_{\pi,c_1.c_2} = \mathcal{I}_{\pi,c}$. $\square$

Conclusion

We have defined the intersection number between a $v-$path and an $e-$path lying on a digital surface, and we have proved that this number of "real" intersections between two paths of surfels is invariant under homotopic deformations of the two paths. The intersection number is a new "topological invariant" in the field of digital surfaces. Then, the intersection number can be used to easily prove a new Jordan theorem for surfels curves (next Chapter). In Chapter 9 we will use this tool in order to prove an important theorem related to the characterization of topology preservation within digital surfaces.

Chapter 7

A new Jordan Theorem

Using the intersection number, we easily prove the following new Jordan theorem.

Theorem 11 *If $\pi = (y_k)_{k=0,\dots,p}$ is a parameterization of a simple closed $n-$curve of surfels on a digital surface Σ, not included in a loop and $n-$reducible in Σ (i.e. $\pi \simeq_n (y_0, y_p)$), then $\Sigma \setminus \pi^*$ has exactly two $\overline{n}-$connected components.*

See Figure 6.7 for an illustration of such simple closed $n-$curves and Figure 6.8 for a counter example (paths which are not reducible).

In the sequel of this chapter, $\pi = (y_k)_{k=0,\dots,p}$ is an $n-$path which satisfies the hypothesis of Theorem 11. We will use the three following lemmas :

Lemma 7.1 *For all $k \in \{0, \dots, p\}$, the sets $Left_\pi(k)$ and $Right_\pi(k)$ are both not empty. Furthermore, y_k is $\overline{n}-$adjacent to a surfel α of $Left_\pi(k)$ and $\overline{n}-$adjacent to a surfel β of $Right_\pi(k)$.*

Proof : Let us suppose that $Left_\pi(k) = \emptyset$, then from Definition 6.2, y_{k-1} and y_{k+1} are $e-$adjacent. Then, y_{k+1} is $e-$adjacent (and so $v-$adjacent) to both y_k and y_{k-1}. Since π has a length greater than 3 (otherwise π^* would be included in a loop) it follows that there exists a surfel $y_j \notin \{y_{k-1}, y_k, y_{k+1}\}$ which is $n-$adjacent to y_{y+1}; and this contradict the fact that π^* is a simple closed $n-$curve. Then, $Left_\pi(k) \neq \emptyset$ and similarly we can prove that $Right_\pi(k) \neq \emptyset$.

If $n = e$ it is immediate that $Right_\pi(k)$ [resp. $Left_\pi(k)$] contain a surfel α [resp. β] which is $v-$adjacent to y_k.

If $n = v$ then we may suppose that no surfel of $Left_\pi(k)$ [resp. $Right_\pi(k)$] is $e-$adjacent to y_k. Then, this implies that y_{k+1} and y_{k-1} belong to the same loop so that c^* is not a simple closed $v-$curve. □

Lemma 7.2 *There exists two surfels α and β in $\Sigma \setminus \pi^*$ which are $\overline{n}-$adjacent to a surfel y_k of π and which are not $\overline{n}-$connected in $\Sigma \setminus \pi^*$.*

Proof : Let α and β be the two surfels defined in Lemma 7.1. Now, we suppose by contraposition the existence of a $\overline{n}-$path $c = (x_i)_{i=0,\dots,q}$ in $\Sigma \setminus \pi^*$ between the surfels α and β, $\overline{n}-$neighbors of the surfel y_k, and which does not intersect π^*. We denote $c' = (x_0 = \alpha, \dots, x_q = \beta, y_k, \alpha)$ which is a closed $\overline{n}-$path. Then, from the definition of the two surfels α and β; and from the very definition of the intersection number between π and c', we have $\mathcal{I}_{\pi,c'} = \pm 1$. Now, since π is $n-$homotopic to a trivial path and from Theorem 9, and also Remark 6.5, we should have $\mathcal{I}_{\pi,c'} = 0$. This contradicts the existence of c (see Figure 7.1 and Figure 7.2). Then, α and β are not $\overline{n}-$connected in $\Sigma \setminus \pi^*$. □

Lemma 7.3 *Let y_k and y_{k+1} be two consecutive surfels of π for $k \in \{0, \dots, p\}$. For any surfel $s \notin \pi^*$, which is $\overline{n}-$adjacent to y_k, there exists a surfel $t \notin \pi^*$, $\overline{n}-$adjacent to y_{k+1}, and a $\overline{n}-$path from s to t in $\Sigma \setminus \pi^*$.*

Proof :

Case when $(n, \overline{n}) = (e, v)$ **:** Then, the lemma can be proved by local considerations. We can suppose that y_{k+1} shares as an edge with y_k the oriented edge $(0, 1)$ of y_k. Then we consider the following cases :

- y_{k-1} shares as an edge with y_k the oriented edge $(1, 2)$ of y_k. Then, we call a the surfel wich shares with y_k its oriented edge $(2, 3)$ and b the surfel which share its oriented edge $(3, 0)$. Neither a nor b can belong to π^* (otherwise, π is not an $e-$curve). Now, for any surfel s of $N_v(y_k) \setminus \pi^*$ we have :
  - $s \in \mathcal{L}_1(y_k)$ so s is $v-$adjacent to $y_{k+1} \in \mathcal{L}_0(y_k)$.
  - $s \in \mathcal{L}_2(y_k)$ so s is $v-$adjacent (or equal) to a which is also $v-$adjacent to b, itself $v-$adjacent to y_{k+1}.
  - $s \in \mathcal{L}_3(y_k)$ so s is $v-$adjacent or equal to b itself $v-$adjacent to y_{k+1}.

- $s \in \mathcal{L}_0(y_k)$ so s is $v-$adjacent or equal to y_{k+1}.

- y_{k-1} shares as an edge with y_k the oriented edge $(2,3)$ of y_k. Then, we call a the surfel wich shares with y_k the oriented edge $(1,2)$ of y_k and b the surfel which shares with y_k the oriented edge $(3,0)$ of y_k. Neither a nor b can belong to π^*. Now, for any surfel s of $N_v(y_k) \setminus \pi^*$ we have :
  - $s \in \mathcal{L}_0(y_k)$ or $s \in \mathcal{L}_1(y_k)$ so s is $v-$adjacent or equal to y_{k+1}.
  - $s \in \mathcal{L}_2(y_k)$ so s is $v-$adjacent (or equal) to a itself $v-$adjacent to y_{k+1}.
  - $s \in \mathcal{L}_3(y_k)$ so s is $v-$adjacent or equal to b itself $v-$adjacent to y_{k+1}.

- y_{k-1} shares with y_k the oriented edge $(3,0)$ of y_k. This case is similar with the first one.

In all cases, we are able to construct a $v-$path in $\Sigma \setminus \pi^*$ from s to a surfel t which is $v-$adjacent to y_{k+1}.

Case when $(n,\overline{n}) = (v,e)$ **:** Since π is a simple closed $v-$curve, then we have $N_v(y_k) \setminus \{y_{k-1}, y_{k+1}\} = N_v(y_k) \setminus \pi^*$. Furthermore, the set $N_v(y_k) \setminus \{y_{k-1}, y_{k+1}\}$ has exactly two $e_{y_k}-$connected components which both contain a surfel $e-$adjacent to y_k (Remark 6.3). Obviously, y_{k+1} is $e-$adjacent to each of the previous $e_{y_k}-$connected components of $\overline{\pi^*}$. Then, any surfel s in one of these connected components is $e-$connected in $\Sigma \setminus \pi^*$ to a surfel t which is $e-$adjacent to y_{k+1} (see Figure 7.3). □

Proof of Theorem 11 : Let y_k be a surfel of π for $k \in \{0, \dots, p\}$. From Lemma 7.2 there exists two surfels α and β in $\overline{\pi^*}$ which are $\overline{n}-$adjacent to the surfel y_k and not $\overline{n}-$connected in $\Sigma \setminus \pi^*$. In particular, $\Sigma \setminus \pi^*$ has at least two $\overline{n}-$connected components. Furthermore, for any surfel $x \in \Sigma \setminus \pi^*$, since Σ is $e-$connected (Theorem 6), then there exists an $\overline{n}-$path c' in $\Sigma \setminus \pi^*$ from x to another surfel x' which is $\overline{n}-$adjacent to a surfel y_h of π^* ($h \in \{0, \dots, p\}$).

Using inductively Lemma 7.3, we see that the surfel x' is $\overline{n}-$connected in $\overline{\pi^*}$ with a surfel x'' of $N_{\overline{n}}(y_k) \setminus \{y_{k-1}, y_{k+1}\}$.

If $n = v$ and since $\mathit{Left}_\pi(k)$ and $\mathit{Right}_\pi(k)$ cannot contain other surfels of π than y_{k-1} and y_{k+1} so that $\mathit{Left}_\pi(k) \cup \mathit{Right}_\pi(k) = N_v(y_k) \setminus \{y_{k-1}, y_{k+1}\}$, then x'' is $e-$connected to either α or β in $\overline{\pi^*}$.

If $n = e$, then no surfel of π other than y_{k-1} and y_{k+1} are $e-$adjacent to y_k. It follows immediately that $N_v(y_k) \setminus \pi^*$ has exactly two $v_{y_k}-$connected components respectively included in $Left_\pi(k)$ and $Right_\pi(k)$. Then, x'' is $v_{y_k}-$connected to either α or β in $\overline{\pi^*}$.

Finally, in both cases, it remains that x is $\overline{n}-$connected to either α or β in $\Sigma \setminus \pi^*$, surfels which are not themselves $\overline{n}-$connected in this set. Then, $\Sigma \setminus \pi^*$ has exactly two $\overline{n}-$connected components. $\Box$

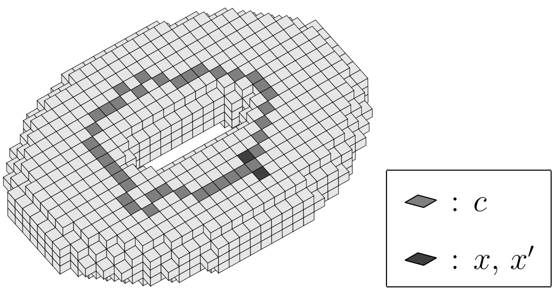


Figure 7.1: The two surfels x and x' may be linked by an $e-$path in $\overline{c^*}$. Then, c cannot be $v-$reducible in Σ.

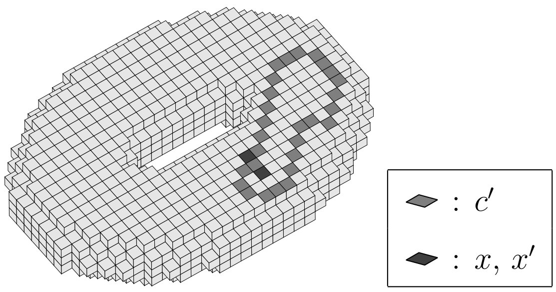


Figure 7.2: Since c' is $v-$reducible in Σ, the two surfels x and x' cannot be $e-$connected in $\overline{c^*}$ (Lemma 7.2).

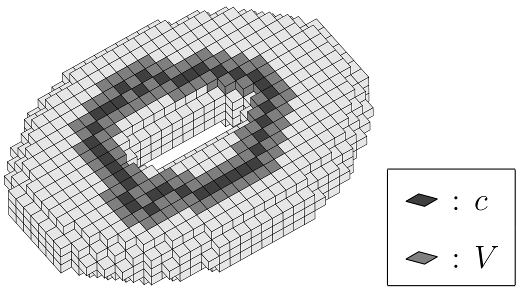


Figure 7.3: Any surfel of the set V (set of surfels of Σ which are $e-$adjacent to c^*) is either $e-$connected in $\Sigma \setminus c^*$ to the surfel x or to the surfel x' of Figure 7.1.

Chapter 8

New properties of the 2D winding number

In this chapter we present a new property of the two dimensional winding number (as defined in Section 2.1.2) which may be deduced from the main property of the intersection number. Indeed, it is stated here that the winding number of a digital closed path c in $\mathbb{Z}^2$ around a pixel $x \notin c^*$ is the same for any closed path c' which is $n-$homotopic to c in $\mathbb{Z}^2 \setminus \{x\}$, for $n \in \{4, 8\}$ (such a property was previously admitted without proof). Indeed, we will define a generalized two dimensional winding number, using the intersection number, so that properties of this new number comes from the main properties of the intersection number. For example, using this new definition, we will also be able to give a straightforward proof of the fact that the winding numbers $W_{x,c}$ and $W_{x',c}$, of an $n-$path c respectively around x and around x', are equal as soon as the two pixels x and x' belong to the same $\overline{n}-$connected component of $\overline{c^*}$. However, we are not interested here by the data of an efficient algorithm which would allow to compute this generalized 2D winding number. Thus, we will only investigate some new properties which are straightforward consequences of the new definition.

Furthermore, the definition of the generalized winding number provides a new definition of the winding number which may be computed using any half $\overline{n}-$path (Definition 8.6) from a pixel x instead of an half straight line from the pixel x considered in the complement of a closed $n-$path c.

First, let us introduce a possible one to one correspondence between any bounded image in $\mathbb{Z}^2$ and the *same* image in a surface Σ. We will use the following definitions.

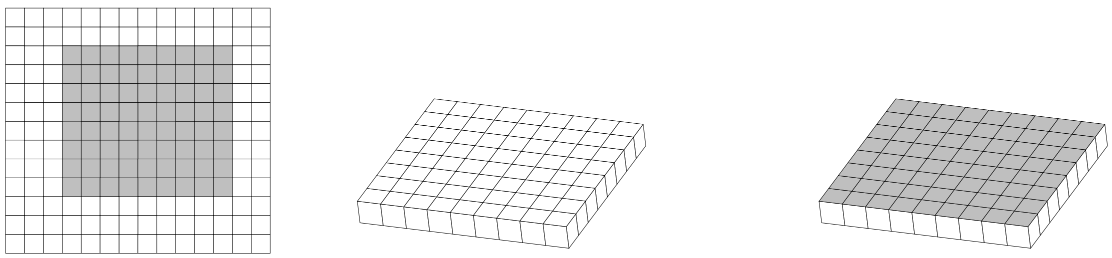
Definition 8.1 **($l\times h-$rectangle)** *Let $(l, h) \in \mathbb{Z}^2$. A subset $\mathcal{R}$ of $\mathbb{Z}^2$ is called an $l\times h-$*rectangle *if there exists $(x, y) \in \mathbb{Z}^2$ such that : $\mathcal{R} = \{\ (x + i, y + j)|\ 1 \leq i \leq l$ and $1 \leq j \leq h\ \}$ (see Figure 8.1(a)). An l×h−rectangle may also be simply called a* rectangle.

Definition 8.2 **($l\times h-$box)** *Let $\mathcal{R}$ be an $l\times h-$rectangle. We define the $l\times h-$*box *$\mathcal{B}$* of $\mathbb{Z}^3$ associated to *$\mathcal{R}$ by : $\mathcal{B} = \{\ (x, y, 0) \in \mathbb{Z}^3|\ (x, y) \in R\}$ (see Figure 8.1(b)). An $l\times h-$box may also be simply called a* box.

Definition 8.3 **($l\times h-$surface)** *Let $\mathcal{R}$ be an $l\times h-$rectangle and $\mathcal{B}$ be the $l\times h-$box associated to $\mathcal{R}$. We define the l×h−*surface *$Sigma_{\mathcal{R}}$* associated to *$\mathcal{R}$ by : $\Sigma_{\mathcal{R}} = \delta_{6+}(B, \overline{B})$ (see Figure 8.1(b)). An $l\times h-$surface may also be simply called a* rectangle.

Finally, we may define the upper part of an $l\times h-$surface :

Definition 8.4 **(upper part of an $l\times h-$surface)** *Let $\mathcal{R}$ be an $l\times h-$rectangle and B be the $l\times h-$box associated to R. Furthermore, let $\Sigma_{\mathcal{R}}$ be the $l\times h-$surface associated to $\mathcal{R}$. We define the* upper part of $\Sigma_{\mathcal{R}}$ *and we denote by $Up(\Sigma_{\mathcal{R}})$ the set of surfels of $\Sigma_{\mathcal{R}}$ of the form $((x, y, 0), (x, y, 1))$ (for all $(x, y) \in \mathcal{R}$). (see Figure 8.1(c)).*



(a) In grey : a $9\times 8-$rectangle in $\mathbb{Z}^2$.

(b) The visible $9\times 8-$surface of a $9\times 8-$box.

(c) In grey : the upper part of a $9\times 8-$surface

Figure 8.1: Illustration of the Definition 8.1-8.4

In the sequel of this chapter, and when no more precision is given, we have $(n, \overline{n}, m, \overline{m}) \in \{(4, 8, e, v), (8, 4, v, e)\}$. Furthermore, $\mathcal{R}$ is an $l\times r-$rectangle of $\mathbb{Z}^2$ and $\Sigma_{\mathcal{R}}$ is the $l\times r-$surface associated to R. We will also denote by Z the upper part of $\Sigma_{\mathcal{R}}$ (i.e. $Z = Up(\Sigma_{\mathcal{R}})$).

Now, we may define the following map.

Definition 8.5 (map $\mathcal{C}$) *We define the one to one correspondance $\mathcal{C}$ between $\mathcal{R}$ and Z by :*

$$\begin{array}{rrcl} \mathcal{C}\ : & \mathcal{R} & \longrightarrow & Z \\ & (x,y) & \longmapsto & ((x,y,0),(x,y,1)) \end{array}$$

The following remark comes immediately :

Remark 8.1 *For any pixel x and y of the rectangle $\mathcal{R}$:*

- *$\mathcal{R}_4(x,y)$ if and only if $\mathcal{R}_e(\mathcal{C}(x),\mathcal{C}(y))$.*
- *$\mathcal{R}_8(x,y)$ if and only if $\mathcal{R}_v(\mathcal{C}(x),\mathcal{C}(y))$.*

Notation 8.1 *For any $n-$path $\pi = (y_k)_{k=0,\dots,p}$ such that $\pi^* \subset \mathcal{R}$, we denote by $\mathcal{C}(\pi)$ the $m-$path π' of Z defined by $\pi' = (\mathcal{C}(y_0),\dots,\mathcal{C}(y_p))$ (this is an $m-$path following Remark 8.1).*

We will also use the following notion of a half path (see Figure 8.2).

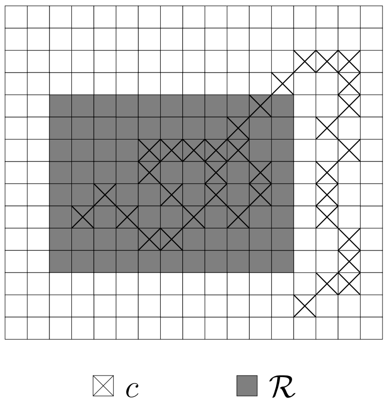
Definition 8.6 (set of half $n-$paths) *An $n-$path $\pi = (y_k)_{k=0,\dots,p}$ with $p > 0$ is called a* half $n-$path of $\mathbb{Z}^2$ from y_0 according to $\mathcal{R}$ *if the two following properties hold :*

i) $y_0 \in \mathcal{R}$.

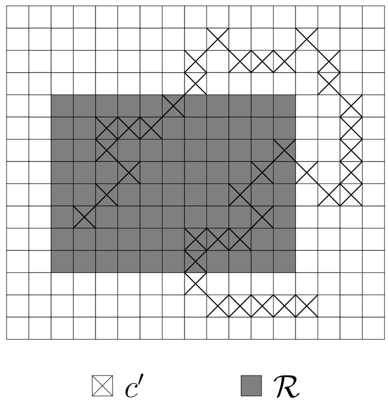
ii) *There exists an integer N (also denoted by $N_\infty(\pi)$) such that $\{y_0,\dots,y_N\} \subset \mathcal{R}$ and for all $k > N$ we have $y_k \notin \mathcal{R}$.*

We denote by $\mathcal{H}_n^x(\mathcal{R})$ the set of half $n-$paths of $\mathbb{Z}^2$ from the pixel x according to $\mathcal{R}$, and by $\mathcal{H}_n(\mathcal{R})$ the union of the sets $\mathcal{H}_n^x(\mathcal{R})$ for all $x \in \mathcal{R}$.

Let x_0 be a surfel of $\overline{Z}$. Now, let z be a surfel of Z which is $e-$adjacent to $\overline{Z}$. One may associate to z exactly one of its (at most two) $e-$neighbors, denoted by $t(z)$, in $\overline{Z}$. Furthermore, one may also associate to the surfel $t(z)$ an arbitrary $e-$path in $\overline{Z}$ from $t(z)$ to x_0 (see Figure 8.3(a)). Finally, we have just "defined" a map G from the border of $\mathcal{R}$ to the set of $e-$paths in $\overline{Z}$ which start at some surfel $t(z)$ $e-$adjacent to a surfel $z \in Z$ and end at x_0. Then, we can define the map $\widetilde{\mathcal{C}}$ which associates to an half $n-$path $\pi = (y_0,\dots,y_p)$ of $\mathcal{H}_n^{y_0}(\mathcal{R})$ (see Figure 8.3(b)) the path π' in Σ defined by : $\pi' = (y_0,\dots,y_N,t(y_N)).G(y_N)$ where $N = N_\infty(\pi)$ (see Figure 8.3(c)).

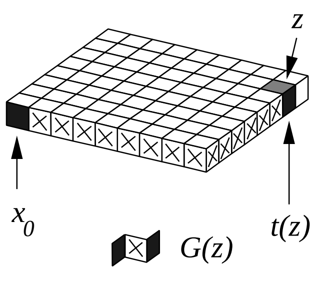


(a) An half 8−path c according to a rectangle $\mathcal{R}$

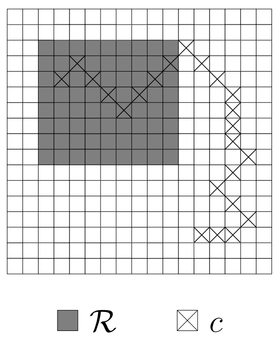


(b) This is not an half path.

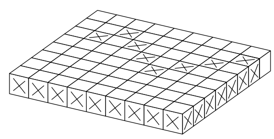
Figure 8.2: Example and counter-example of half paths according to a rectangle $\mathcal{R}$.



(a) The path $G(z)$.



(b) An 8−path c in $\mathbb{Z}^2$ which exits from a rectangle $\mathcal{R}$.



(c) The path $\widetilde{\mathcal{C}}(c)$.

Figure 8.3: Illustration of the construction of a map $\widetilde{\mathcal{C}}$ which associates an n−path in $\mathcal{R}$ with an m−path in $\Sigma_{\mathcal{R}}$.

The following proposition comes with no need of a proof; indeed, it is simply a straightforward consequence of the definition of the two dimensional winding number (Section 2.1.2) and the definition of the intersection number (Section 6.1).

Proposition 8.1 *Let c be a closed $n-$path in $\mathcal{R}$ and x be a pixel of $\mathcal{R} \setminus c^*$. Furthermore, let $\pi' = \widetilde{\mathcal{C}}(\Delta_x^\alpha)$ for $\alpha \in \{1,2,3,4\}$, then $W_{x,c}^\alpha = \mathcal{I}_{\pi',\mathcal{C}(c)}$ (see Section 2.1.1 for the definition of the half straight lines Δ_x^α).*

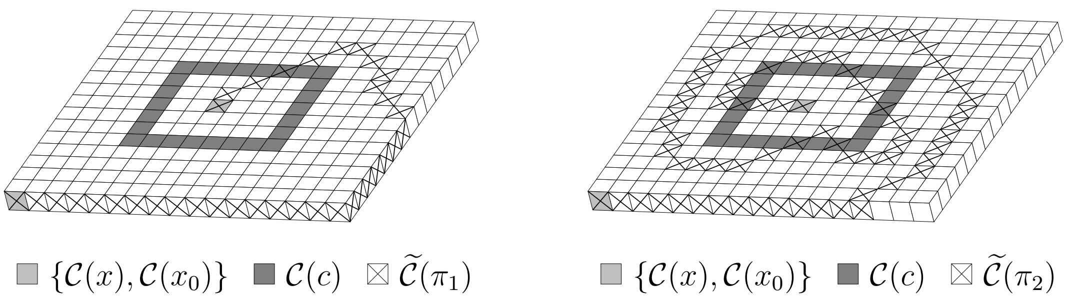
The following remark comes from Proposition 2.5 and the fact that $\Sigma_\mathcal{R}$ is obviously simply $n-$connected for $n \in \{e, v\}$ (the reader who needs a proof can prove it from $\chi_n(\Sigma_\mathcal{R}) = 2$ and the definition of a disk).

Remark 8.2 *Let c be a closed $\overline{n}-$path in $\mathcal{R}$ and x be a pixel of $\mathcal{R} \setminus c^*$. If π_1 and π_2 are two half $n-$paths of $\mathcal{H}_n^x(\mathcal{R})$, then $\widetilde{\mathcal{C}}(\pi_1)$ and $\widetilde{\mathcal{C}}(\pi_2)$ are $m-$homotopic.*

Now, we also state the following proposition which will allow to justify Definition 8.7.

Proposition 8.2 *Let c be a closed $\overline{n}-$path in $\mathcal{R}$ and x be a pixel of $\mathcal{R} \setminus c^*$. If π_1 and π_2 are two half $n-$paths of $\mathcal{H}_n^x(R)$ then $\mathcal{I}_{\widetilde{\mathcal{C}}(\pi_1),\mathcal{C}(c)} = \mathcal{I}_{\widetilde{\mathcal{C}}(\pi_2),\mathcal{C}(c)}$.*

Proof : From Remark 8.2, we have $\widetilde{\mathcal{C}}(\pi_1) \simeq_m \widetilde{\mathcal{C}}(\pi_2)$ in Σ. Now, the properties $\mathcal{P}(\widetilde{\mathcal{C}}(\pi_1), \mathcal{C}(c))$, $\mathcal{P}(\widetilde{\mathcal{C}}(\pi_2), \mathcal{C}(c))$, $\mathcal{P}(\mathcal{C}(c), \widetilde{\mathcal{C}}(\pi_1))$ and $\mathcal{P}(\mathcal{C}(c), \widetilde{\mathcal{C}}(\pi_2))$ obviously hold so that from Theorem 10 we have $\mathcal{I}_{\widetilde{\mathcal{C}}(\pi_1),\mathcal{C}(c)} = \mathcal{I}_{\widetilde{\mathcal{C}}(\pi_2),\mathcal{C}(c)}$ (see Figure 8.4). $\square$



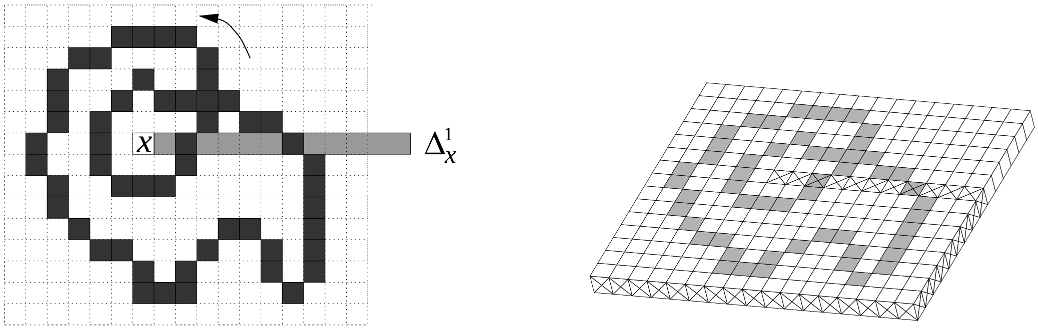
(a) $\mathcal{I}_{\widetilde{\mathcal{C}}(\pi_1),\mathcal{C}(c)} = \pm 1$ (b) $\mathcal{I}_{\widetilde{\mathcal{C}}(\pi_2),\mathcal{C}(c)} = \pm 1$

Figure 8.4: Illustration of Proposition 8.2

We observe that an half straight line Δ_x^α from x (see Section 2.1.1) belongs to $\mathcal{H}_x^n(\mathcal{R})$ for $n \in \{4, 8\}$ as depicted by Figure 8.5(a). It follows immediately that the following definition of a new two dimensional winding number coincides with the definition of Section 2.1.2 in the particular case when π is an half straight line (see Figure 8.5).

Definition 8.7 (generalized 2D winding number) *Let c be a closed $n-$path in $\mathbb{Z}^2$ such that $c^* \subset \mathcal{R}$. We define the* generalized winding number of c around x, *denoted by $\Omega_{x,c}$, as follows : $\Omega_{x,c} = \mathcal{I}_{\widetilde{\mathcal{C}}(\pi),\mathcal{C}(c)}$ where π is any $\overline{n}-$path of $\mathcal{H}_n^x$.*

The generalized two dimensional winding number is well defined since it effectively does not depend on the choice of the half $\overline{n}-$path π following Proposition 8.2. This allows to state the following theorem which is a restriction of the Theorem 9 in the context of 2D images (see Figure 8.6).



(a) A $15 \times 17-$rectangle $\mathcal{R}$, a pixel x and an $8-$path c included in $\mathcal{R} \setminus \{x\}$. The half line Δ^1_x is an half $4-$path according to $\mathcal{R}$. $W^1_{x,c} = 1$.

(b) $c' = \mathcal{C}(c)$ and $\pi' = \widetilde{\mathcal{C}}(\Delta^1_x)$. $\mathcal{I}_{\pi',c'} = 1$.

Figure 8.5: The classical 2d winding number and the generalized one.

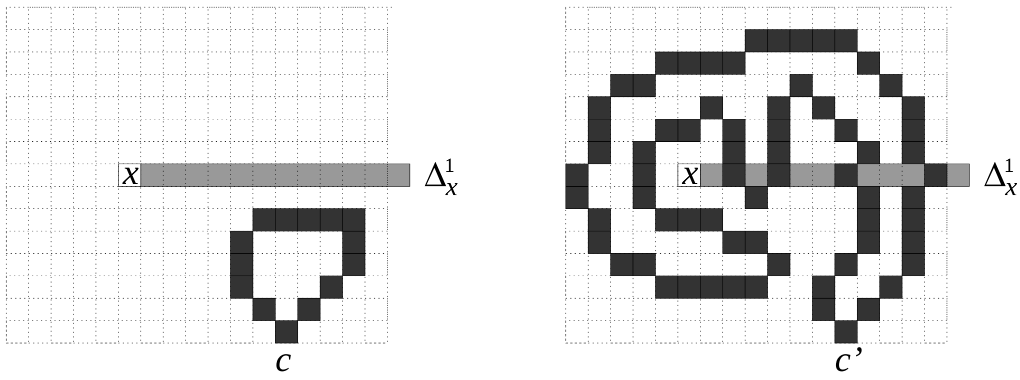
Now, we can state the following theorem.

Theorem 12 *Let $x \in \mathbb{Z}^2$ and let c and c' be two closed $n-$paths which are $n-$homotopic in $\mathcal{R} \setminus \{x\}$. Then, $W_{x,c} = W_{x,c'}$.*

Proof : From Definition 8.7 we have $W_{x,c} = \mathcal{I}_{\widetilde{\mathcal{C}}(\Delta^1_x),\mathcal{C}(c)}$ and $W_{x,c'} = \mathcal{I}_{\widetilde{\mathcal{C}}(\Delta^1_x),\mathcal{C}(c')}$. From the very definition of the map $\mathcal{C}$, is is immediate that $\mathcal{C}(c) \simeq_m \mathcal{C}(c')$ in $Z\{\mathcal{C}(x)\}$. Then, since $\mathcal{P}(\widetilde{\mathcal{C}}(\Delta^1_x), \mathcal{C}(c))$ and $\mathcal{P}(\widetilde{\mathcal{C}}(\Delta^1_x), \mathcal{C}(c'))$ obviously hold it comes from Theorem 9 that $\mathcal{I}_{\widetilde{\mathcal{C}}(\Delta^1_x),\mathcal{C}(c)} = \mathcal{I}_{\widetilde{\mathcal{C}}(\Delta^1_x),\mathcal{C}(c')}$. $\square$

Observe that the latter theorem may be easily generalized to any homotopic deformation of the paths in $\mathbb{Z}^2 \setminus \{x\}$ instead of $\mathcal{R} \setminus \{x\}$.

The following proposition was proved by R. Malgouyres in [58] using a less immediate proof because of the *restriction* imposed by the use of an half straight line in the definition of $W_{x,c}$. Now, this proposition is a straightforward consequence of the definition of $\Omega_{x,c}$.



(a) (b)

Figure 8.6: The 2d winding number $W_{x,c}$ is left unchanged by any 8−homotopic deformation of c in $\mathbb{Z}^2 \setminus \{x\}$.

Proposition 8.3 ([58]) *Let x be a pixel of $\mathbb{Z}^2$ and c be a closed $n-$path in $\mathbb{Z}^2 \setminus \{x\}$. If x' is pixel of $\overline{c^*}$ which is $\overline{n}-$connected to x in $\overline{c^*}$ then $\Omega_{x,c} = \Omega_{x',c}$.*

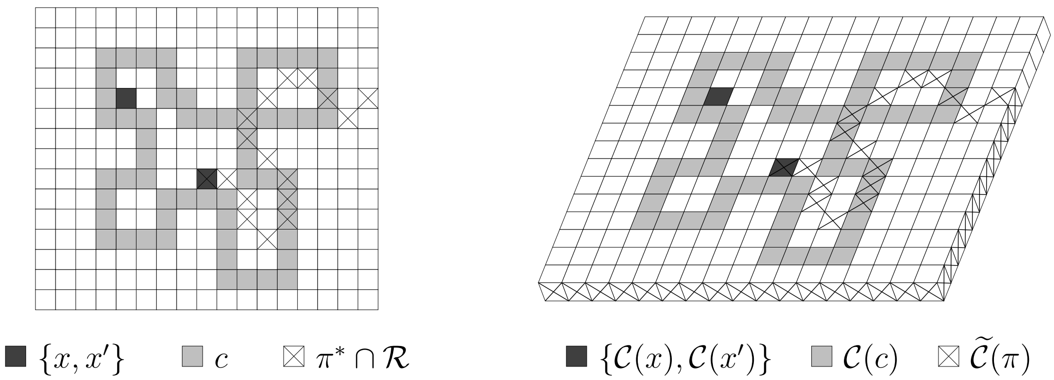
Sketch of proof (different from [58]) : Let $\mathcal{R}$ be a rectangle of $\mathbb{Z}^2$ such that $c^* \cup \{x, x'\} \subset \mathcal{R}$ and also such that x and x' are $\overline{n}-$connected in $\mathcal{R} \setminus c^*$. We have $\Omega_{x,c} = \mathcal{I}_{\widetilde{\mathcal{C}}(\pi),\mathcal{C}(c)}$ where π is an half $\overline{n}-$path from x related to $\mathcal{R}$. Then, let γ be an $\overline{n}-$path from x' to x in $\mathcal{R} \setminus c^*$. Since $x \notin c^*$ and $\gamma^* \cap c^* = \emptyset$, then it is immediate from the definition of the intersection number that $\mathcal{I}_{\widetilde{\mathcal{C}}(\pi),\mathcal{C}(c)} = \mathcal{I}_{\widetilde{\mathcal{C}}(\gamma.\pi),\mathcal{C}(c)}$. Now, the $\overline{n}-$path $\gamma.\pi$ is an half $\overline{n}-$path from x' according to $\mathcal{R}$ so that $\mathcal{I}_{\widetilde{\mathcal{C}}(\gamma.\pi),\mathcal{C}(c)} = \Omega_{x',c}$. Finally, $\Omega_{x',c} = \Omega_{x,c}$.
$\square$

In Figure 8.7(a) we have depicted a closed 4−path c in a rectangle $\mathcal{R}$ and an half 8−path π from a pixel x (dark pixel with a cross) which allows to compute the generalized winding number of c around x. Following Definition 8.7, we use the corresponding paths depicted in Figure 8.7(b). Then the two following properties hold :

– The two dark pixels x and x' are 8−connected in $\mathcal{R}$ by a 8−path γ which does not intersect c.

– The two dark surfels $\widetilde{\mathcal{C}}(x)$ and $\widetilde{\mathcal{C}}(x')$ are $v-$connected in $Up(\Sigma_{\mathcal{R}})$ by a $v-$path $\widetilde{\mathcal{C}}(\gamma)$ which does not intersect $\widetilde{\mathcal{C}}(c)$.

Then, the path $\gamma.\pi$ is an half 8−path from x' according to $\mathcal{R}$ and it is obvious that $\mathcal{I}_{\widetilde{\mathcal{C}}(\gamma.\pi),\widetilde{\mathcal{C}}(c)} = \mathcal{I}_{\widetilde{\mathcal{C}}(\pi),\widetilde{\mathcal{C}}(c)}$.



(a) (b)

Figure 8.7: The generalized winding number of the closed 8−path c (figure (a)) is the same around each dark pixels.

Chapter 9

New characterization of topology preservation

9.1 A new theorem about homotopy in digital surfaces

In Chapter 5 we have given the definition of homotopy between subsets of a digital surface Σ. In this section, we are interested by the characterization of homotopy which involves the digital fundamental group (see Sections 2.3 and 5.3). Then, using the intersection number as defined in Chapter 6, we state and prove a new characterization of topology preservation.

The purpose of this section is to prove that the condition "i_* is an isomorphism" of Theorem 8 is sufficient to say that each $\overline{n}-$connected component of $\overline{Y}$ contains a surfel of $\overline{X}$, except in the very particular case when X is the whole surface Σ which is a topological sphere (see Definition 5.9) and Y is a disk obtained by removing from X a topological disk (Definition 5.8). In other words, except in the above mentioned particular case, condition 2 of Theorem 8 is in fact implied by condition 1 of Theorem 8.

In other words, we prove the following theorem:

Theorem 13 *Let $Y \subset X \subset \Sigma$ be two $n-$connected sets such that $X \neq \Sigma$ or Σ is not a sphere or $X \setminus Y$ is not a topological disk, or Y is not a topological disk, then:*
Y is lower $n-$homotopic to X if and only if the morphism $i_* : \Pi_1^n(Y, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i : Y \longrightarrow X$ is an isomorphism for any base surfel $B \in Y$.

To prove this theorem, we suppose that some (X, Y) satisfies the condition 1 of Theorem 8, but does not satisfies condition 2 of Theorem 8. In other words, we suppose that i_* is an isomorphism for any base surfel $B \in Y$ and we also suppose the existence of an $\overline{n}-$connected component of $\overline{Y}$ which contains no point of $\overline{X}$. Namely, we suppose the existence of an $\overline{n}-$connected component of $\overline{Y}$, denoted by A, such that $A \subset X$. In a first step, we prove that this $\overline{n}-$connected component A is a topological disk. In a second step, we will show by an indirect way that the set $X \setminus A$ is a topological disk too, in fact equal to Y, and conclude that $X = \Sigma$ and X is a sphere.

In the sequel of this section, $Y \subset X$ are two $n-$connected subsets of a digital surface Σ, and we suppose that for any surfel $B \in Y$, the group morphism i_* between $\Pi_1^n(Y, B)$ and $\Pi_1^n(X, B)$ induced by the inclusion map of Y in X, is an isomorphism, as in Theorem 8. In further proof, we will use the following simple Lemma.

Lemma 9.1 *Let $Y \subset X$ be two $n-$connected subsets of Σ and B be a surfel of Y. Then, the two following properties are equivalent :*

i) The morphism $i_* : \Pi_1^n(Y, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion of Y in X is an isomorphism.

ii) For all surfels B' in Y, the morphism $i'_* : \Pi_1^n(Y, B') \longrightarrow \Pi_1^n(X, B')$ induced by the inclusion of Y in X is an isomorphism.

Proof : We only have to prove that property $i)$ implies property $ii)$. Suppose that $i_* :\ \Pi_1^n(Y, B) \longrightarrow \Pi_1^n(X, B)$ is group isomorphism and let B' be any surfel of Y. Then, let i'_* be the group morphism from $\Pi_1^n(Y, B')$ to $\Pi_1^n(X, B')$ induced by the inclusion of Y in X. Now, following Proposition 2.10, let i_Y and i_X be the two canonical group isomorphisms respectively from $\Pi_1^n(Y, B)$ to $\Pi_1^n(Y, B')$ and from $\Pi_1^n(X, B)$ to $\Pi_1^n(X, B')$. Clearly, we have $i'_* = i_X \circ i_* \circ i_Y^{-1}$ so that i'_* is an isomorphism. $\square$

9.2 First step of the proof

In this section, A is a connected component of $\overline{Y}$ which contains no surfel of $\overline{X}$ (i.e. $A \subset X$) and B is a surfel of Y (B is the base surfel of the digital fundamental groups which are considered in this section).

Lemma 9.2 *There exists a surfel $x_0 \in A$ such that the morphism :*

$$i''_* : \Pi_1^n(Y \cup (A \setminus \{x_0\}), B) \longrightarrow \Pi_1^n(X, B)$$

induced by the inclusion map $i'' : Y \cup (A \setminus \{x_0\}) \longrightarrow X$ is an isomorphism and Y is lower $n-$homotopic to $Y \cup (A \setminus \{x_0\})$.

Corollary 9.3 *The set $\{x_0\}$ is $\overline{n}-$homotopic to A, so that A is a topological disk for the $\overline{n}-$adjacency relation.*

Before to prove Lemma 9.2, we have to prove two preliminary results.

Lemma 9.4 *Let x_0 be a surfel of A. If A is composed of at least 2 surfels, then there exists a surfel $x \neq x_0$ in A which is $n-$adjacent to Y and which is $n-$simple for $Y \cup \{x\}$.*

Corollary 9.5 *Let x_0 be a surfel of A. If A is composed of at least 2 surfels, then there exists a surfel $x \neq x_0$ in A which is $n-$adjacent to Y and which is $\overline{n}-$simple for A.*

Proof : From Lemma 9.4, there exists a surfel $x \neq x_0$ in A which is $n-$adjacent to Y and $n-$simple for $Y \cup \{x\}$. Then, x is neither $n-$isolated in $Y \cup \{x\}$ nor $n-$interior to $Y \cup \{x\}$ (since $A \subset \overline{Y}$ is $\overline{n}-$connected and $x \neq x_0$). Then, following Remark 5.1 and since x is $n-$simple for $Y \cup \{x\}$ we have $Card(\mathcal{C}_n^x[G_n(x, Y \cup \{x\})]) = Card(\mathcal{C}_{\overline{n}}^x[G_{\overline{n}}(x, \overline{Y \cup \{x\}})]) = 1$. Now, since $x \in A$ cannot be $\overline{n}-$adjacent to any other $\overline{n}-$connected component of $\overline{Y}$ than A, it follows that $\mathcal{C}_{\overline{n}}^x[G_{\overline{n}}(x, \overline{Y \cup \{x\}})] = \mathcal{C}_{\overline{n}}^x[G_{\overline{n}}(x, A)]$. Now, since x is $n-$adjacent to Y it is not $\overline{n}-$interior to A then x is $\overline{n}-$simple for A. $\square$

Proof of Lemma 9.4 : Let x be a surfel of $A \subset \overline{Y}$ which is $n-$adjacent to Y and whose distance to x_0 is maximal among all surfels of A which are $n-$adjacent to Y. The distance used here is the length of a shortest $n-$path in A between two surfels. Let us prove that the surfel x is $n-$simple for $Y \cup \{x\}$. We have $G_n(x, Y \cup \{x\}) = G_n(x, Y)$ and $G_{\overline{n}}(x, \overline{Y} \setminus \{x\}) = G_{\overline{n}}(x, \overline{Y})$. Suppose that this surfel x is not $n-$simple for $Y \cup \{x\}$. Since x is neither $n-$isolated nor $n-$interior to $Y \cup \{x\}$, this implies that $Card(\mathcal{C}_n^x(G_n(x, Y))) = Card(C_{\overline{n}}^x(G_{\overline{n}}(x, \overline{Y}))) \geq 2$ (Remark 5.1). Let a and b be two surfels $n-$adjacent to x in two distinct n_x-connected components of $G_n(x, Y)$ which are $n-$adjacent to x.

Let us denote by π_0 the $n-$path (b, x, a). Since a and b are $n-$adjacent to x and do not belong to the same n_x-connected component of $N_v(x) \cap Y$, a is not n_x-adjacent to b.

Following Remark 6.3 it follows that none of the sets $Left_{\pi_0}(1)$ and $Right_{\pi_0}(1)$ is empty and each one contains a surfel which is $\overline{n}-$adjacent to x. Furthermore, if we suppose that all the surfels of $Left_{\pi_0}(1)$ or $Right_{\pi_0}(1)$ which are $\overline{n}-$adjacent to x belong to Y, it is immediate that the two surfels a and b are n_x-connected in $N_v(x) \cap Y$. Then, there must exists two surfels s_1 and s_2 which are $\overline{n}-$adjacent to x such that $s_1 \in Right_{\pi_0}(1) \cap A$ and $s_2 \in Left_{\pi_0}(1) \cap A$. Moreover, we may assume that s_1 and s_2 are $n-$adjacent to Y. Since the set Y is $n-$connected, there exists an $n-$path β_1 from a to B in Y and an $n-$path β_2 from B to b in Y.

Now, an $\overline{n}-$path $\alpha_1 = (s_1, \ldots, x_0)$ in $A \setminus \{x\}$ from s_1 to the surfel x_0 must exist since the $\overline{n}-$distance between x and x_0 is maximal among all surfels of A which are $n-$adjacent to Y. Indeed, otherwise, let c be a shortest $\overline{n}-$path with a length l between x and x_0 in A. If s_1 is not $\overline{n}-$connected to x_0 in $A \setminus \{x\}$ and since s_1 is $\overline{n}-$adjacent to x, then s_1 is at a distance of $l+1$ from x_0 in A. This contradicts the fact that x is at a maximal distance l from x_0 among all the surfels of A which are $n-$adjacent to Y. Similarly, there must exist an $\overline{n}-$path $\alpha_2 = (s_2, \ldots, x_0)$ from s_2 to x_0 in $A \setminus \{x\}$.

Let α be the closed $\overline{n}-$path $\alpha = (x).\alpha_1.\alpha_2^{-1}.(x)$ in A. Note that, from the very construction of α_1 and α_2, we have $x \notin \alpha_1^*$ and $x \notin \alpha_2^*$. We can also construct a closed $n-$path $\beta = (B).\beta_2.\pi_0.\beta_1.(B)$ from B to B in $Y \cup \{x\}$ with $\pi_0 = (b, x, a)$ and $x \notin \beta_1^* \cup \beta_2^*$ since β_1 and β_2 are $n-$paths in Y. We deduce that the two paths α and β only cross each other one time in x, and since $s_1 \in Right_{\pi_0}(1)$ and $s_2 \in Left_{\pi_0}(1)$, we have $\mathcal{I}_{\alpha,\beta} = -\mathcal{I}_{\beta,\alpha} = 1$.

Now, since the morphism i_* from $\Pi_1^n(Y,B)$ to $\Pi_1^n(X,B)$ induced by the inclusion of Y in X is an isomorphism. In particular, i_* is onto and then, for any equivalence class $[c']_{\Pi_1^n(X,B)}$, there exists a closed $n-$path $c \in A_n^B(Y)$ which is $n-$homotopic to c' in X so that $i_*([c]_{\Pi_1^n(Y,B)}) = [c']_{\Pi_1^n(X,B)}$. In our case, there exists an $n-$path $\gamma \in A_n^B(Y)$ which is $n-$homotopic to the $n-$path β in X and $i_*([\gamma]_{\Pi_1^n(Y,B)}) = [\beta]_{\Pi_1^n(X,B)}$. If γ is $n-$homotopic to β in X, and from Theorem 9, we deduce that $\mathcal{I}_{\alpha,\beta} = \mathcal{I}_{\alpha,\gamma} = 1$. But since α is an $\overline{n}-$path in $A \subset \overline{Y}$ and γ is an $n-$path in Y, we have $\gamma^* \cap \alpha^* = \emptyset$ and then $\mathcal{I}_{\alpha,\gamma} = 0$ and we obtain a contradiction. Finally, the point x must be $n-$simple for $Y \cup \{x\}$. $\square$

Remark 9.1 *If x is an $n-$simple surfel for $Y \cup \{x\}$, then, since x is $\overline{n}-$simple in A the set $A \setminus \{x\}$ is $\overline{n}-$connected.*

Proof of Lemma 9.2 : By induction of Lemma 9.4 (and using Lemma 5.1) we show that there exists a sequence of surfels $(s_0, \ldots, s_l)$ such that for all $i \in \{0, \ldots, l\}$, $s_i \in A$ is $n-$simple for $Y \cup \{s_0, \ldots, s_i\}$ and $A \setminus \{s_0, \ldots, s_l\} = \{x_0\}$. Therefore, Y

is lower $n-$homotopic to $Y \cup (A \setminus \{x_0\})$. From Lemma 5.1, the morphism $i^i_* : \Pi^n_1(Y \cup \{s_0, \ldots, s_{i-1}\}, B) \longrightarrow \Pi^n_1(Y \cup \{s_0, \ldots, s_i\}, B)$ induced by the inclusion of $Y \cup \{s_0, \ldots, s_{i-1}\}$ in $Y \cup \{s_0, \ldots, s_i\}$ is a group isomorphism. On the other hand, the morphism $i'_* : \Pi^n_1(Y, B) \longrightarrow \Pi^n_1(Y \cup \{s_i | i = 0, \ldots, l\} = Y \cup (A \setminus \{x_0\}), B)$ induced by the inclusion map $i' : Y \longrightarrow Y \cup (A \setminus \{x_0\})$ is such that $i'_* = i^l_* \circ \ldots \circ i^0$. Therefore, the morphism i'_* is an isomorphism. Furthermore, since i_* is an isomorphism, then $i''_* = i_* \circ {i'_*}^{-1}$ is an isomorphism from $\Pi^n_1(Y \cup (A \setminus \{x_0\}), B)$ to $\Pi^n_1(X, B)$. $\square$

9.3 Second step of the proof

In Section 9.2 we have proved that Y is lower $n-$homotopic to $Y \cup (A \setminus \{x_0\})$ where x_0 is an isolated surfel of $\overline{Y \cup (A \setminus \{x_0\})}$. In this section, we will state that, under the condition that the $n-$path surrounding $\{x_0\}$ in $Y \cup (A \setminus \{x_0\})$ is $n-$reducible in $Y \cup (A \setminus \{x_0\})$, then $Y \cup (A \setminus \{x_0\})$ is a topological disk.

9.3.1 Edgel borders of a connected subset $X \subset \Sigma$

First, we have to define explicitly what we call a "border" of a connected set of surfels. Let X be an $n-$connected subset of a surface Σ.

Definition 9.1 (border edgel) *We call a* border edgel *of X any couple (x, y) of surfels of Σ such that $x \in X$ and $y \in \overline{X}$. We denote by $\mathcal{B}(X)$ the set of border edgels of X.*

Definition 9.2 ($s-$adjacency relation) *We say that two border edgels (x, y) and (x', y') of $\mathcal{B}(X)$ are $s-$*adjacent *if the three following conditions are satisfied :*

- *x, y, x' and y' belong to a common loop $\mathcal{L}$ of Σ.*
- *$x \neq x'$ or $y \neq y'$.*
- *x is $e-$connected to x' in $\mathcal{L} \cap X$ if $n = e$, and y is $e-$connected to y' in $\mathcal{L} \cap \overline{X}$ if $n = v$.*

We can define the $s-$connectivity between border edgels as the transitive closure of this adjacency relation. The definition of an $s-$path of border edgels also comes immediately. Note that any $s-$connected component of border edgels of X is a simple closed curve (i.e.

each border edgel has exactly two $s-$neighbors, one per loop which contains this border edgel) and is called a border *of X, whereas a parameterization of such a simple closed curve is called a* parameterized border of X.

Definition 9.3 ($n-$path $c_n(s)$) *Let $s = (s_0 = (x_0, y_0), \ldots, s_l = (x_l, y_l))$ be a $s-$path of border edgels of X. We define the $n-$*path associated with *s denoted by $c_n(s)$ according to the following cases :*

- *If $n = e$ and for $i \in \{0, \ldots, l-1\}$, we call c_i the shortest $e-$path joining x_i to x_{i+1} in $X \cap \mathcal{L}$, where $\mathcal{L}$ is the unique loop containing $\{x_i, x_{i+1}, y_i, y_{i+1}\}$ (path which exists according to the definition of the $s-$adjacency between s_i and s_{i+1}). Then $c_e(s) = c_0 * \ldots * c_{l-1}$.*
- *If $n = v$ and for $i \in \{0, \ldots, l-1\}$, then x_i is $v-$adjacent to x_{i+1}. We define $c_v(s) = (x_0, \ldots, x_l)$.*

Remark 9.2 *For any $s-$path of border edgels $s = (s_0 = (x_0, y_0), \ldots, s_l = (x_l, y_l))$ of $X \subset \Sigma$ and $n \in \{e, v\}$, all the surfels of $c_n(s)$ are $\overline{n}-$adjacent to $\overline{X}$.*

9.3.2 Free group

In the following, we will use the notion of the (non-Abelian) *free group with m generators*. Let $\mathcal{A} = \{a_1, \ldots, a_m\} \cup \{a_1^{-1}, \ldots, a_m^{-1}\}$ be an alphabet with $2m$ distinct letters, and let $\mathcal{W}_m$ be the set of all words over this alphabet (i.e. finite sequences of letters of the alphabet). We say that two words $w \in \mathcal{W}_m$ and $w' \in \mathcal{W}_m$ are *the same up to an elementary cancellation* if one can be obtained by inserting or deleting in the other a sequence of the form $a_i^{-1}a_i$ or a sequence of the form $a_i a_i^{-1}$ with $i \in \{1, \ldots, m\}$. Now, two words $w \in \mathcal{W}_m$ and $w' \in \mathcal{W}_m$ are said to be *free equivalent* if there is a finite sequence $w = w_1, \ldots, w_k = w'$ of words of $\mathcal{W}_m$ such that for $i = 2, \ldots, k$ the words w_{i-1} and w_i are the same up to an elementary cancellation. This defines an equivalence relation on $\mathcal{W}_m$ and we denote by $\mathcal{F}_m$ the set of equivalence classes of this equivalence relation. Furthermore, if w is a word of $\mathcal{W}_m$, we denote by $\overline{w}$ the equivalence class of the word w following the latter equivalence relation. The concatenation of words defines an operation on $\mathcal{F}_m$ which provides $\mathcal{F}_m$ with a group structure (we define $\overline{w_1}\,\overline{w_2} = \overline{w_1 w_2}$). The group thus defined is called the *free group* with m generators over $\mathcal{A}$. Classically, we denote by $w_1.w_2$ the word obtained by concatenation of the words w_1 and w_2.

We denote by 1_m the unit element of $\mathcal{F}_m$ which is equal to $\overline{\epsilon}$ where ϵ is the empty word. The only result which we shall admit on the free group is the classical result that if a word $w \in \mathcal{L}_m$ is such that $\overline{w} = 1_m$ and w is not the empty word, then there exists in w two successive letters $a_i a_i^{-1}$ or $a_i^{-1} a_i$ with $i \in \{1, \dots, m\}$. This remark leads to an immediate algorithm to decide whether a word $w \in \mathcal{W}_m$ is such that $\overline{w} = 1_m$ by successive cancellations.

9.3.3 Free group element associated with a path

In the sequel $X = \{x_1, \dots, x_l\}$ is an $n-$connected subset of Σ with cardinality $l > 1$.

Notation 9.1 If x is a surfel of X, we abbreviate and denote by $o(x)$ the cardinality of $\mathcal{C}_n^x[G_n(x, X)]$, set of n_x-connected components of $N_v(x) \cap X$ which are $n-$adjacent to x. We observe that $o(x_i)$ may be at most equal to 4 . Then, we may assign a number t in $\{1, \dots, o(x_i)\}$ to each element of $\mathcal{C}_n^{x_i}[G_n(x_i, X)]$ so that it makes sense to talk of the t^{th} element of $\mathcal{C}_n^{x_i}[G_n(x_i, X)]$.

Definition 9.4 (alphabet $\mathcal{A}_X$) *Now, we define the alphabet $\mathcal{A}_X$ as follows :*

$$\begin{aligned} \mathcal{A}_X &= \{x_{1,1}, \dots, x_{1,o(x_1)}, x_{2,1}, \dots, x_{2,o(x_2)}, \dots\dots, x_{l,1}, \dots, x_{l,o(x_l)}\} \\ &\cup \{x_{1,1}^{-1}, \dots, x_{1,o(x_1)}^{-1}, x_{2,1}^{-1}, \dots, x_{2,o(x_2)}^{-1}, \dots\dots, x_{l,1}^{-1}, \dots, x_{l,o(x_l)}^{-1}\} \end{aligned}$$

Where the symbols $\{x_{i,1}, \dots, x_{i,o(i)}\}$ are associated to the surfel x_i, and the symbol $x_{i,j}$ is associated to the j^{th} element of $\mathcal{C}_n^{x_i}[G_n(x_i, X)]$.

Definition 9.5 (word associated to a path) *If $\pi = (y_0, y_1)$ is an $n-$path in X with a length 1 such that $y_0 = x_a$ and $y_1 = x_b$ for $\{a, b\} \subset \{0, \dots, l\}$. We associate to π a word $w_n(\pi, X)$ of the alphabet $\mathcal{A}_X$ defined by $w_n(\pi, X) = x_{a,t} x_{b,u}^{-1}$ where t and u are such that x_b belongs to the t^{th} element of $\mathcal{C}_n^{x_a}[G_n(x_a, X)]$ and x_a belongs to the u^{th} element of $\mathcal{C}_n^{x_b}[G_n(x_b, X)]$.*

If $\pi = (y_k)_{k=0,\dots,p}$ is an $n-$path with a length $q > 1$ in X, we define the word $w_n(\pi, X)$ as follows :

$$w_n(c, X) = w_n((y_0, y_1), X) w_n((y_1, y_2), X) w_n((y_2, y_3), X) \dots w_n((y_{p-1}, y_p), X)$$

And we define $w_n(\pi, X)$ to be the empty word if π is of length 0 or is a trivial path.

Definition 9.6 (free group element associated to a path) *If π is an $n-$path in X and $\mathcal{A}_X$ has a cardinality of $2m$. We define the element $\nu_n(\pi, X)$ of the free group with m generators over $\mathcal{A}_X$ by : $\nu_n(\pi, X) = \overline{w_n(\pi, X)}$.*

Remark 9.3 *If π_1 and π_2 are two $n-$paths in X such that the last surfel of π_1 is equal to the first surfel of π_2. Then, $\overline{w_n(\pi_1.\pi_2, X)} = \overline{w_n(\pi_1, X)w_n(\pi_2, X)}$*

Remark 9.4 *If π is an $n-$path in X, then, from its very construction, $w_n(\pi, X)$ cannot contain some pair $x_{i,t}x_{i,t}^{-1}$ for $i \in \{0, \dots, l\}$ and $t \in \{1, \dots, o(x_i)\}$.*

Proposition 9.6 *Let X be an $n-$connected subset of Σ with at least two surfels. Let π and π' be two $n-$paths in X. If $\pi \simeq_n \pi'$ then $\nu(\pi, X) = \nu(\pi', X)$.*

The proof of Proposition 9.6 relies on the three following lemmas.

Lemma 9.7 *If π is an $n-$back an forth in X, then $\nu_n(\pi, X) = 1_{2m}$.*

Proof : Let $\pi = (y_0, y_1, y_0)$ be an $n-$back and forth in X such that $y_0 = x_a$ and $y_1 = x_b$. From Definition 9.6, $w_n(\pi, X) = x_{a,u}x_{b,t}^{-1}x_{b,t}x_{a,u}^{-1}$ if y_1 belongs to the t^{th} element of $\mathcal{C}_n^{y_0}[G_n(y_0, X)]$; and y_0 belongs to the u^{th} element $\mathcal{C}_n^{y_1}[G_n(y_1, X)]$. Finally, it is immediate that $\overline{w_n(\pi, X)} = 1_{2m}$. $\square$

Lemma 9.8 *If π is a triplet in X, then $\nu_v(\pi, X) = 1_{2m}$.*

Proof : Let $\pi = (y_0, y_1, y_2, y_0)$ be a triplet in X. Then, we may suppose without loss of generality (up to a new numbering of X) that $y_0 = x_0$, $y_1 = x_1$ and $y_2 = x_2$. Since y_0, y_1 and y_0 belong to a common loop, the surfels y_1 and y_2 belong to the same element of $\mathcal{C}_v^{y_0}[G_v(y_0, X)]$ (say the first one, still without loss of generality); the two surfels y_0 and y_2 belong to the same (say the second) element of $\mathcal{C}_v^{y_1}[G_v(y_1, X)]$; and the two surfels y_0 and y_1 belong to the same (say the third) element of $\mathcal{C}_v^{y_2}[G_v(y_2, X)]$. Thus, $w_v((y_0, y_1), X) = x_{0,1}x_{1,2}^{-1}$, $w_v((y_1, y_2), X) = x_{1,2}x_{2,3}^{-1}$, and $w_v((y_2, y_0), X) = x_{2,3}x_{0,1}^{-1}$ so that $w(\pi, X) = x_{0,1}x_{1,2}^{-1}x_{1,2}x_{2,3}^{-1}x_{2,3}x_{0,1}^{-1}$. Then,
$\overline{w(\pi, X)} = \overline{x_{0,1}x_{1,2}^{-1}x_{1,2}x_{2,3}^{-1}x_{2,3}x_{0,1}^{-1}} = \overline{x_{0,1}x_{2,3}^{-1}x_{2,3}x_{1,0}^{-1}} = \overline{x_{1,0}x_{1,0}^{-1}} = 1_{2m}$. $\square$

Lemma 9.9 *If π is an $e-$loop in X, then $\nu_e(\pi, X) = 1_{2m}$.*

Proof : Let $\pi = (y_0, \ldots, y_p)$ be an $e-$loop in X. First, we observe that $p > 2$ and we may suppose that $y_i = x_i$ for all $i \in \{0, \ldots, p\}$. Then, from Definition 9.5 we have $w_v(\pi, X) = w_v((x_0, x_1), X).w_v((x_1, x_2), X). \ldots .w_v((x_{p-1}, x_p), X)$.

Furthermore, for all $k \in \{1, \ldots, p-1\}$ let us denote by $\sigma(k)$ the number of the element of $\mathcal{C}_v^{x_k}[G_v(x_k, X)]$ which contains the surfel x_{k-1}. Then, from the very definition of an $e-$loop in X, it is immediate that $\sigma(k)$ is also the number of the element of $\mathcal{C}_v^{x_k}[G_v(x_k, X)]$ which contains the surfel x_{k+1} (indeed, x_{k-1} and x_{k+1} are both $e-$connected in $\pi^* \subset N_v(x_k) \cap X$). On the other hand, it is also obvious that x_1 and x_{p-1} both belong to the same element of $\mathcal{C}_v^{x_0}[G_v(x_0, X)]$, say the first one. It follows that :

$$w_v(\pi, X) = x_{0,1} x_{1,\sigma(1)}^{-1} x_{1,\sigma(1)} x_{2,\sigma(2)}^{-1} x_{2,\sigma(2)} x_{3,\sigma(2)}^{-1} \cdots x_{p-1,\sigma(p-1)}^{-1} x_{p-1,\sigma(p-1)} x_{0,1}^{-1}$$

And then : $\overline{w_v(\pi, X)} = \overline{x_{0,1} x_{0,1}^{-1}} = 1_{2m}$. □

Proof of Proposition 9.6 :

Following Proposition 6.24 and Proposition 6.17, it is sufficient to prove this proposition in the case when π and π' are the same up to an elementary $\mathcal{T}-$deformation when $n = v$ and the same up to an elementary $\mathcal{S}-$deformation when $n = e$.

If $n = e$ we suppose that $\pi = \pi_1.(s).\pi_2$ and $\pi' = \pi_1.\gamma.\pi_2$ where γ is an $e-$back and forth or an $e-$loop in X. Then, following Remark 9.3 we have $\nu_e(\pi, X) = \overline{w_e(\pi_1, X) w_e(\pi_2, X)}$ and $\nu_e(\pi', X) = \overline{w_e(\pi_1, X) w_e(\gamma, X) w_e(\pi_2)}$. Now, from Lemma 9.7 and Lemma 9.9, we have $\overline{w_e(\gamma, X)} = 1_{2m}$ and it follows that $\overline{w_e(\pi_1, X) w_e(\gamma, X) w_e(\pi_2, X)} = \overline{w_e(\pi_1, X) w_e(\pi_2, X)}$. Finally, $\nu_e(\pi, X) = \nu_e(\pi', X)$.

If $n = v$ we suppose that $\pi = \pi_1.(s).\pi_2$ and $\pi' = \pi_1.\gamma.\pi_2$ where γ is a $v-$back and forth or a triplet in X. Then, following Remark 9.3 we have $\nu_v(\pi, X) = \overline{w_v(\pi_1, X) w_v(\pi_2, X)}$ and $\nu_v(\pi', X) = \overline{w_v(\pi_1, X) w_v(\gamma, X) w_v(\pi_2, X)}$. Now, from Lemma 9.7 and Lemma 9.8, we have $\overline{w_n(\gamma, X)} = 1_{2m}$ and it follows that $\overline{w_v(\pi_1, X) w_v(\gamma, X) w_v(\pi_2, X)} = \overline{w_v(\pi_1, X) w_v(\pi_2, X)}$. Finally, $\nu_v(\pi, X) = \nu_v(\pi', X)$. □

9.3.4 Important lemmas

The main result of this section is constituted by the following proposition :

Proposition 9.10 *Let Y be an $n-$connected subset of Σ and x_0 be an $\overline{n}-$isolated surfel of $\overline{Y}$ (i.e. x_0 has no $\overline{n}-$neighbor in $\overline{Y}$). Let s be the $s-$curve $((a, x_0), (b, x_0), (c, x_0), (d, x_0))$*

where a, b, c and d are the appropriately named four $e-$neighbors of x_0 in Y. If $c_n(s)$ is $n-$homotopic to a trivial path in Y, then Y is a topological disk.

In the sequel of this section, Y is an $n-$connected subset of Σ and s is a parameterized border of Y (i.e. s is a parameterization of a simple closed $s-$curve of border edgels of Y). In order to prove Proposition 9.10, we must state the following lemmas.

Lemma 9.11 *If the $n-$path $c_n(s)$ is $n-$homotopic in Y to a trivial path and $(c_n(s))^*$ has more than one surfel then $c_n(s)$ contains a surfel which is $n-$simple for Y.*

In order to prove Lemma 9.11, we first state the following lemma.

Lemma 9.12 *If $w_n(c_n(s), X)$ contains a pair $x_{i,k}^{-1}x_{i,k}$ for some i in $\{1, \dots, l\}$ and some k in $\{1, \dots, o(x_j)\}$ then the surfel c^k of $c_n(s)$ such that $x_i = c^k$ is $n-$simple for X.*

Proof : If $x_{j,b}^{-1}x_{j,b}$ occurs in $w_n(c_n(s), X)$, then more precisely and from Definition 9.5, $x_{k,u}x_{j,b}^{-1}x_{j,b}x_{k',t}$ occurs for some k and k' in $\{1, \dots, l\}$, u in $\{1, \dots, o(x_k)\}$ and t in $\{1, \dots, o(x_{k'})\}$. It means that there exists in $c_n(s)$ a subsequence $(c^p, \dots, c^{p+q})$ such that :

- $w_n((c^p, \dots, c^{p+q}), X) = (w_n((c^p, c^{p+1}), X) \dots w_n((c^{p+q-1}, c^{p+q}), X) = x_{j,b}^{-1}x_{j,b}$.
- $c^p = x_k$ belongs to the b^{th} element of $\mathcal{C}_n^{x_j}[G_n(x_j, X)]$, and x_j belongs to the u^{th} element of $\mathcal{C}_n^{x_k}[G_n(x_k, X)]$.
- $c^{p+q} = x_{k'}$ belongs to the b^{th} element of $\mathcal{C}_n^{x_j}[G_n(x_j, X)]$; and x_j belongs to the t^{th} element of $\mathcal{C}_n^{x_{k'}}[G_n(x_{k'}, X)]$.
- $c^k = x_j$ for all $k \in \{p+1, \dots, p+q\}$

In other words, the parameterized border comes from an $n_{_i}-$connected component of $G_n(x_i, X)$ to x_i and exits from x_i to the same $n_{x_i}-$connected component of $G_n(x_i, X)$. It is then immediate that $G_n(x_i, X)$ has a single $n_{x_i}-$connected component (see Figure 9.1) $n-$adjacent to x_i which is itself $\overline{n}-$adjacent to a surfel of $\overline{X}$ (Remark 9.2). Then, x_i is $n-$simple for X. $\square$

Proof of Lemma 9.11 : Since $c_n(s)$ is closed and has a length greater then 1 it follows that $w_n(c_n(s), X)$ is a word on $\mathcal{A}_X$ with a length (number of symbols) greater or equal to 4 (see Definition 9.5). Now, since $c_n(s)$ is $n-$homotopic to a trivial path in

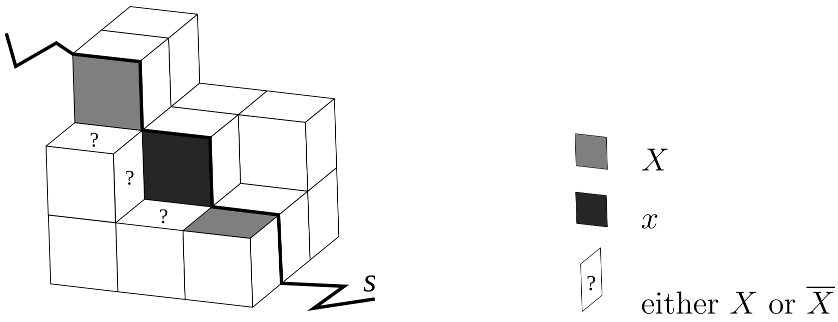
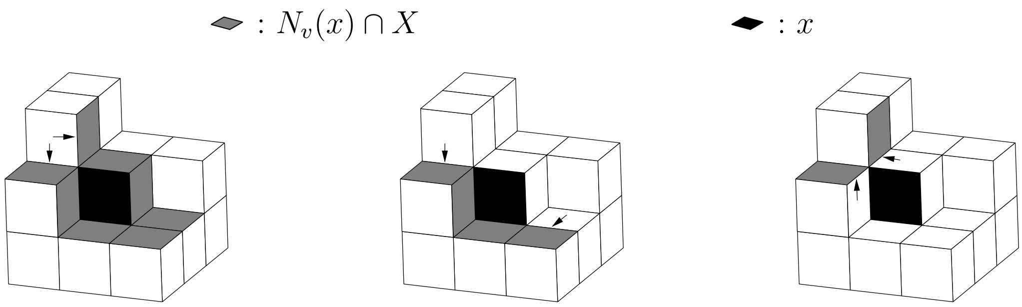


Figure 9.1: If the two grey surfels belong to the same v_x-connected of $G_v(x, X)$ it is clear that $Card(\mathcal{C}_v^x[G_v(x, X)]) = 1$ and $Card(\mathcal{C}_e^x[G_e(x, \overline{X})]) = 1$.

Y, it follows from Proposition 9.6 and Definition 9.5 for a word associated to a trivial path, that $\overline{w_n(c_n(s), X)} = 1_{2m}$. Then, $w_n(c_n(s), X)$, having a length greater than 1, must necessarily contain a pair $x_{j,b}^{-1}x_{j,b}$ associated to a surfel x_j and the b^{th} $n_{x_j}-$connected component of $G_n(x_j, X)$ $n-$adjacent to x_j. Indeed, from the very definition of the word $w_n(c_n(s), X)$, no pair of the form $x_{j,b}x_{j,b}^{-1}$ can occur in this word for any $j \in \{0, \ldots, l\}$. But, from Lemma 9.11, this implies that $c_n(s)$ contains a surfel which is $n-$simple for X. □

Definition 9.7 (border edgels associated with an element of $\mathcal{C}_n^x[G_n(x, X)]$) *Let x be an $n-$simple surfel of $Z \subset \Sigma$, and let C and D be the only elements of respectively $\mathcal{C}_n^x[G_n(x, Z)]$ and $\mathcal{C}_{\overline{n}}^x[G_{\overline{n}}(x, \overline{Z})]$ (see Definition 5.3 and Remark 5.1). The two edgels of the form (a, b) and (a', b'), where $\{a, a'\} \subset C$ and $\{b, b'\} \subset D$, are called the* two border edgels associated to the component C *(see Figure 9.2).*



(a) For $(n, \overline{n}) = (e, v)$ (b) $(n, \overline{n}) = (e, v)$ or (v, e) (c) $(n, \overline{n}) = (v, e)$

Figure 9.2: Border edgels associated with an n_x-connected component of $G_n(x, X)$.

Lemma 9.13 *Let x be a surfel of $c_n(s)$ which is $n-$simple for Y. Let f and f' be the two border edgels associated with the unique connected component N_0 of $G_n(x,Y)$. Let s' be a parameterized border of $Y \setminus \{x\}$ which contains the two border edgels f and f' and such that $c_n(s')$ is a path from x' to x' where $x' \neq x$. Then the surfel path $c_n(s')$ is reducible in $Y \setminus \{x\}$.*

Sketch of proof : First, $c_n(s)^*$ must have more than 1 surfel since x is simple. It follows that it is possible to find an edgel path s_2 such that $c_n(s)$ and $c_n(s_2)$ are the same up to a change of parameter but the extremity of $c_n(s_2)$ is different from x. Following Lemma 2.6, $c_n(s_2)$ is reducible too.

Now, let s' be the $s-$path obtained by removing in s_2 the edgels between f and f' which contain x (maybe such edgels do not exist as in the case of Figure 9.2(a)) and possibly replacing them with edgels of the form (x,q) where q belongs to N_0. If $f=(b,d)$ and $f'=(b',d')$ then let γ be the sub-path of $c_n(s_2)$ from b to b' associated with the $s-$path from f to f' in s_2, and let γ' be the sub-path from b to b' of $c_n(s')$ associated with the $s-$path from f to f' in s'. These two paths have the same extremities and are included in $C \cup \{x\}$ where C is the only n_x-connected component of $\mathcal{C}_n^x[G_n(x,Y)]$. Then, it is easily seen that γ is $n-$homotopic to γ' in Y, so that the paths $c_n(s_2)$ and $c_n(s')$ are $n-$homotopic too. It follows that $c_n(s_2)$ is reducible in Y. $\Box$

Proof of Proposition 9.10 : We show the existence of a sequence of deletion of $n-$simple surfels which leads to $\{y\}$ from Y where y is a surfel of Y.

Let $s^0 = ((a,x_0),(b,x_0),(c,x_0),(d,x_0))$ be the $s-$curve of Proposition 9.10 and we set $Y^0 = Y$. Now, if $m \geq 0$ and if s^m is a parameterized border of a set Y^m with at least 2 surfels such that $c_n(s^m)$ is $n-$homotopic to the trivial path in Y^m, then, Lemma 9.11 shows that $c_n(s^m)$ contains an $n-$simple surfel for Y^m which we denote by y^m. So, let N_0^m be the connected component of $G_n(y^m, Y^m)$ $n-$adjacent to y^m and let f and f' be the two border edgels associated with N_0^m. Then, let $Y^{m+1} = Y^m \setminus \{y^m\}$ and s^{m+1} be the parameterized border of Y^{m+1} which contains the two border edgels f and f' as defined in Lemma 9.13, and let b^{m+1} be the basepoint of $c_n(s^{m+1})$ (distinct from y^m following Lemma 9.13).

From Lemma 9.13, the path $c_n(s^m)$ is $n-$homotopic to $c_n(s^{m+1})$ in Y^m and $c_n(s^{m+1})$ is reducible in Y^m.

Now, let $i_*^m : \Pi_1^n(Y^{m+1}, b^{m+1}) \longrightarrow \Pi_1^n(Y^m, b^{m+1})$ be the morphism induced by the inclusion of Y^{m+1} in Y^m. Since y^m is $n-$simple for Y, Lemma 5.1 implies that the morphism i_*^m is a group isomorphism, in particular i_*^m is one to one.

Then, $i_*^m([c_n(s^{m+1})]_{\Pi_1^n(Y^{m+1},b^{m+1})}) = [c_n(s^{m+1})]_{\Pi_1^n(Y^m,b^{m+1})}$ but since the path $c_n(s^{m+1})$ is $n-$reducible in Y^m, it follows that $i_*^m([c_n(s^{m+1})]_{\Pi_1^n(Y^{m+1},b^{m+1})}) = [1]_{\Pi_1^n(Y^m,b^{m+1})}$. On the other hand, we have $i_*^m([1]_{\Pi_1^n(Y^{m+1},b^{m+1})}) = [1]_{\Pi_1^n(Y^m,b^{m+1})}$. Then, since i_*^m is one to one we obtain that $[1]_{\Pi_1^n(Y^{m+1},b^{m+1})} = [c_n(s^{m+1})]_{\Pi_1^n(Y^{m+1},b^{m+1})}$. In other words, the $n-$path $c_n(s^{m+1})$ is reducible in Y^{m+1}, so Y^{m+1} and s^{m+1} still satisfy conditions of Lemma 9.11.

By induction on the integer m we prove that while the set Y^m has more than two surfels, we can find a surfel $y^m \in Y^m$ which is $n-$simple for Y^m and so a set $Y^{m+1} = Y^m \setminus \{y^m\}$ which is lower $n-$homotopic to Y^m and strictly included in Y^m. Finally, there must exist an integer k such that Y^k is reduced to a single surfel $\{y\}$. It is clear from its construction that $Y^k = \{y\}$ is lower $n-$homotopic to $Y^0 = Y$, so that Y is a topological disk. $\square$

9.4 Proof of Theorem 13

Proof of Theorem 13 :

We use Theorem 8 and prove that Condition 2 is implied by Condition 1 except in a very particular case. So, we suppose that Condition 1 is satisfied and that there exists a $\overline{n}-$connected component A of $\overline{Y}$ which is included in X (i.e. A contains no surfel of $\overline{X}$).

From Lemma 9.2 there exists a surfel x_0 in A such that Y is lower $n-$homotopic to $Y \cup (A \setminus \{x_0\})$. Then, since i_* and $i''_* :\ \Pi_1^n(Y, B) \longrightarrow \Pi_1^n(Y \cup (A \setminus \{x_0\}), B)$ are isomorphisms for all $B \in Y$, the group morphism $i'_* : \Pi_1^n(Y \cup (A \setminus \{x_0\}), B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i' :\ Y \cup (A \setminus \{x_0\}) \longrightarrow X$ satisfying $i_* = i'_* \circ i''_*$ is an isomorphism. Since x_0 belongs to the $\overline{n}-$connected component A of $\overline{Y}$, the surfel x_0 is an $\overline{n}-$isolated surfel of $\overline{Y \cup (A \setminus \{x_0\})}$ and let s be a parameterization of the border between $Y \cup (A \setminus \{x_0\})$ and $\{x_0\}$. Let $c_n(s)$ be the $n-$path associated with the $s-$path s.

First, we suppose that $c_n(s)$ is not reducible in $Y \cup (A \setminus \{x_0\})$. It is clear that the same path $c_n(s)$ is reducible in $Y \cup A$ and so in X. Thus, let z_0 be the base surfel of $c_n(s)$ and j_* be the morphism from $\Pi_1^n(Y \cup (A \setminus \{x_0\}), z_0)$ to $\Pi_1^n(X, z_0)$ induced by the inclusion of $Y \cup (A \setminus \{x_0\})$ in X. Then, j_* cannot be one to one and from Lemma 9.1 the morphism i'_* cannot be an isomorphism since B and z_0 are $n-$connected in $Y \cup (A \setminus \{x_0\})$. It follows that i_* is not an isomorphism and we get a contradiction.

Therefore $c_n(s)$ is $n-$homotopic to the path reduced to a single surfel in $Y \cup (A \setminus \{x_0\})$. Then, by Proposition 9.10 we know that $Y \cup (A \setminus \{x_0\})$ is a topological disk. From Lemma 9.2, Y is lower $n-$homotopic to $Y \cup (A \setminus \{x_0\})$ so we have $\chi_n(Y) = \chi_n(Y \cup (A \setminus \{x_0\}) = 1$. Since $Y \neq \Sigma$, the set Y is a topological disk (Definition 5.8). Condition 2 of Lemma 5.2 shows that $\overline{Y}$ has a single $\overline{n}-$connected component so $\overline{Y} = A$. From Lemma 9.2 it is straightforward that $\{x_0\}$ is lower $\overline{n}-$homotopic to A, so that A is a topological disk. Then, $Y \cup A = \Sigma$ and $Y \cup A \subset X \subset \Sigma$ so $X = \Sigma$. Since $\chi_n(Y) = 1$ and $\chi_{\overline{n}}(\overline{Y}) = \chi_n(A) = 1$, then $\chi_n(\Sigma) = \chi_n(Y) + \chi_{\overline{n}}(\overline{Y}) = 2$. This ends to prove that Condition 1 of Theorem 8 is implied by Condition 2 of Theorem 8 except in the particular case when X is the whole surface Σ which is a sphere and $\overline{Y}$ is a topological disk as well as Y. And we obtain Theorem 13. $\square$

Conclusion of Part II

The intersection number, which was initially used in order to prove a basic Jordan theorem for digital curves lying on a digital surface (see [32]), has been used here among other tools to prove that the fundamental group can be used to completely characterize lower homotopy between subsets of a digital surface. Thus, the intersection number appears as a good tool for proving theorems of topology within digital surfaces.

Now, we have achieved to show that topology preservation within digital surfaces is strictly related to properties involving the digital fundamental groups of objects. The framework of digital surfaces appears as an intermediate framework for digital topology between the 2D and 3D digital spaces. However, characterizing the lower homotopy between subsets of $\mathbb{Z}^3$ is still a difficult and open problem. Indeed, the digital fundamental group is not sufficient in this case.

Thus, the results of this part show that digital surfaces constitute an interesting and fruitful field of investigations, intermediate step between the 2D and the 3D cases.

It is interesting to observe that the possible numbers of *real* intersections between closed curves drawn on a closed surface of $\mathbb{R}^3$ is related to the genus of the surface. Indeed, it is for example possible to draw two curves which intersect only once on the surface of a solid torus whereas this is impossible on a sphere. The intersection number, which has been defined here, allows such considerations in the digital field.

Part III

A contribution to the study of 3D digital topology

Introduction to Part III

The digital fundamental group, as introduced by Kong in [45], involves equivalence classes of paths according to a relation of deformation for digital closed paths. It is an important tool in the field of digital topology and in particular, it is used as a criterion of topology preservation for 3D digital objects (see [49],[8] and [29]). Now, the question remains about the existence of an efficiently computable characterization of the lower homotopy between objects of $\mathbb{Z}^3$. Such a difficult question cannot be solved today because of the lack of theoretical tools for studying the topology of three dimensional discrete objects. In particular, we should provide new tools dedicated to the study of homotopy classes of discrete paths.

Several authors have been studying homotopy classes of paths in 2D. Rosenfeld and Nakamura in [93] have, among other things, established the relation between 2D holes and the fact that two curves can or cannot be deformed one into each other. In [66], Malgouyres gives an algorithm to decide whether two closed paths in 2D are homotopic or not. In [32] and [31] we have introduced a new tool which helps in distinguishing homotopy classes of paths lying on the surface of a 3D object, and which have been presented in the previous part. One purpose of this new part is to provide sufficient conditions under which a discrete closed path in an object $X \subset \mathbb{Z}^3$ cannot be deformed in X into another one.

More precisely, we introduce an analogue to the *linking number* of simple closed curves in $\mathbb{R}^3$ defined in classical topology and knot theory (see [85]). Intuitively, the linking number counts the number of times a given closed path is interlaced with another one. This linking number has the same properties as its continuous analogue. A very intuitive one is that it is left unchanged when one of the considered paths is *continuously* deformed, i.e by an homotopic deformation in the sense of Definition 2.11, in the complement of the other. Furthermore, as a step of the proof of the latter property, we also prove that the linking number well behaves with respect to concatenation of paths. In other words, the linking number between the concatenation of two closed path and a third one is nothing but the sum of their linking numbers with the third one. Because of its invariance property, the linking number can be practically and formally used to distinguish two homotopy classes of paths as soon as one can find a path which does not have the same linking number with two elements, one in each of the considered classes.

Since the digital linking number is expected to be invariant under homotopic deformation of the paths in the complement of each other, it will be defined for paths following the classical duality for adjacencies. Clearly, two closed and linked $26-$paths can be unlinked by an homotopic deformation of one in the complement of the other whereas this cannot occur between a $26-$path and a $6-$path (with the associated relations of deformation).

Note that in this part, we chose not to consider the continuous analogues of the discrete paths in order to prove the main properties of the linking number. On the other hand, the proofs given here for the main theorems are self sufficient and only use the basic notions classically defined in the field of digital topology and which have been recalled in Part I.

Furthermore, this linking number leads to an intuitive proof of the fact that the number of tunnels in an object $X \subset \mathbb{Z}^3$ is strictly related to the number of tunnels in its complement (this is the subject of Chapter 12). Indeed, the number of tunnels mentioned here must be understood as the number given by the computation of the Euler characteristic. In this case, the equality between the two numbers is immediate. However, the localization of the tunnels is not provided by the Euler characteristic. This fact will be illustrated in Section 10.1. Then, a solution to this drawback consists in the use of the digital fundamental group. But a link between tunnels of an object and tunnels of its complement is then difficult to state and the linking number will help in this case. Thus, in Chapter 12, we will prove that a new and concise characterization of 3d simple voxels may be given using the digital fundamental group. This characterization is said to be new since it does not involve the digital fundamental group of the complement the object in order to characterize the fact that a voxel is simple. Indeed, it is shown that the new characterization is equivalent to the classical one, although it has one less condition.

Chapter 10

Previous characterizations of topology preservation

In this chapter, we explain how several authors have defined simple voxels (i.e. simple spels in $\mathbb{Z}^3$). Indeed, there exists two main ways to define topology preservation by deletion of a voxel in an object of $\mathbb{Z}^3$. The first one involves the Euler characteristic whereas the second uses the digital fundamental group. Both methods come from algebraic topology and have been adapted to digital topology.

As introduced in Section 3.1, continuous deformations can be simulated in digital space by sequences of elementary local deformations (i.e. addition or deletion of points) which are admitted as topologically preserving. Thus, it is important to rigorously establish what means *topology preservation* when one deals with the deletion or addition of points. Within objects of $\mathbb{Z}^3$, in addition to connectedness considerations as in the 2D case (see Definition 3.6), we must also take care to preserve the *tunnels* of the object. Tunnels are those *things* which distinguish the two objects depicted in Figure 5.1 of the introduction to Chapter 5. Now, there are two formal ways to define tunnels as already mentioned in the previous part. The Euler characteristic can be used to count their number in an object but shows its insufficiency to fully characterize topology preservation. The second one, the digital fundamental group, uses the fact that a tunnel in an object of $\mathbb{Z}^3$, as well as in a digital surface, can be detected by the existence of a closed path which is not reducible in this object. In the two following sections, we will recall both tools.

10.1 Using the Euler characteristic

10.1.1 Definition of $\chi_n(X)$

Given a subset P of $\mathbb{R}^3$, we say that P is *polyhedral* if P can be seen as the union of points, closed straight line segments, closed triangles and closed tetrahedra. In the context of *cellular complexes*, points, segments, triangles and tetrahedra are called $d-$*cells* with a respective dimension d equal to 0, 1, 2 and 3. Now, the Euler characteristic of such an object, denoted by $\chi(P)$, is the following alternated sum (see [49] or [71]) :

$$\chi(P) = \text{number of } 0-\text{cells - nb. of } 1-\text{cells} + \text{nb. of } 2-\text{cells - nb. of } 3-\text{cells} \qquad (10.1)$$

In Figure 10.1 we have depicted several polyhedral sets with the corresponding alternated sum.

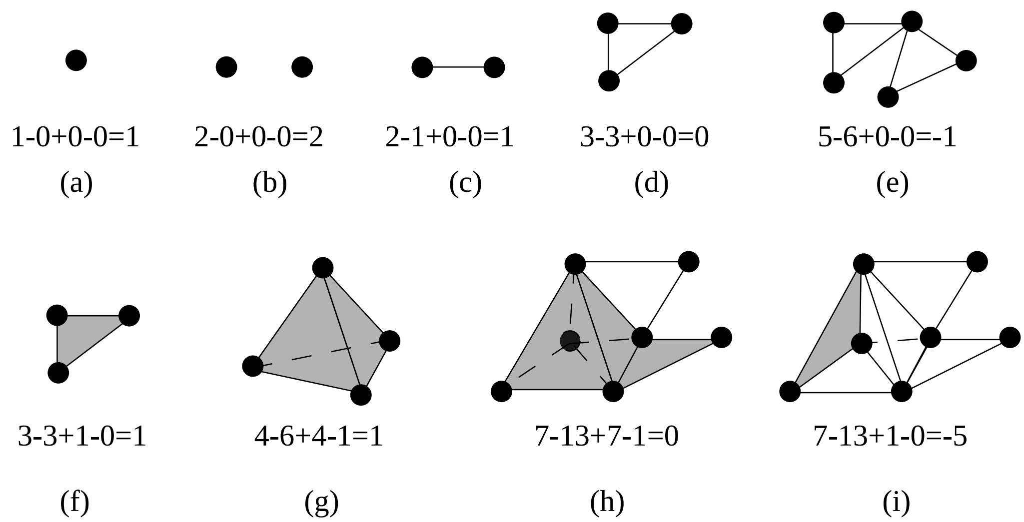
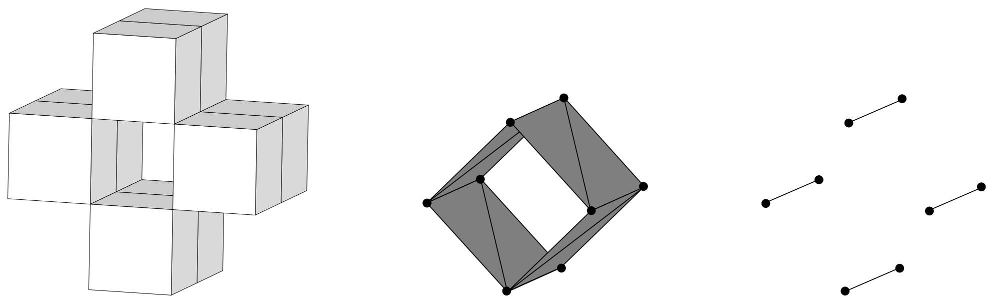


Figure 10.1: Example of Euler characteristic of several polyhedral sets, and the corresponding alternated sums.

Then, it can be seen that the Euler characteristic of a set P is also equal to the number of connected components of P plus its number of cavities minus the number of tunnels in P. For example, the set depicted in Figure 10.1(e) has two tunnel since it is connected and has no cavities. Now, some methods have been developed by several authors to compute the Euler characteristic of a digital object. This Euler characteristic uses the notion of a *polyhedral continuous analogue* of a digital object. Obviously, the way to construct such continuous analogue is very dependent to the couple of adjacencies considered. Thus, the continuous analogue of the object depicted in Figure 10.2(a) could be as depicted in

Figure 10.2(b) for $(n, \overline{n}) = (26, 6)$ whereas it may be like in Figure 10.2(c) if $(n, \overline{n}) = (6, 26)$. See [48] for details about the way such a continuous analogue can be constructed. If we denote by $C_n(X)$ the continuous analogue of a digital object X, then $\chi_n(X)$, the *Euler characteristic of* X, is defined by $\chi_n(X) = \chi(C_n(X))$.



(a) An object $X \subset \mathbb{Z}^3$ (b) $C_{26}(X)$ (c) $C_6(X)$

Figure 10.2: An object X of $\mathbb{Z}^3$ and its two possible continuous analogues.

Now, provided a way to compute the Euler characteristic of a digital object by considering its continuous analogue, and since the number of connected components of the object and its complement are easily computable; it becomes possible to give the *number of tunnels* in a digital object of $\mathbb{Z}^3$. Then, such defined tunnels can be used to give some local characterization of simple voxels. However, we will see that the Euler characteristic does not allow us to give an acceptable *global* characterization of simple points, though it is sufficient to provide a local one.

10.1.2 First characterization of simple voxels

Definition 10.1 (simple voxel : ambiguous definition) *Let* $X \subset \mathbb{Z}^3$ *and* $x \in X$. *The voxel* x *is* $n-$*simple for* X *if the four following conditions are satisfied :*

i) X *and* $X \setminus \{x\}$ *have the same number of* $n-$*connected components.*

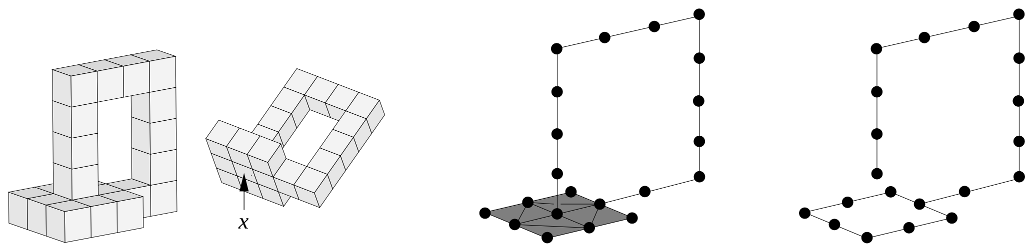
ii) $\overline{X}$ *and* $\overline{X} \cup \{x\}$ *have the same number of* $\overline{n}-$*connected components.*

iii) *No tunnel of* X *is removed or created by deletion of* x.

iv) *No tunnel of* $\overline{X}$ *is removed or created by addition of* x.

This first definition will be clarified in the sequel, and we will see that it can be locally characterized using connectivity considerations in some particular neighborhoods (subsection 10.2), this in an analogous way to the case of simple pixels in two dimensions. In fact, the restriction of Definition 10.1 to the case of two dimensional images leads to the definition of a simple pixel (Definition 3.6). Indeed, the removal or the creation of a tunnel in a subset of $\mathbb{Z}^2$ respectively implies the merging of two connected components of the background or the creation of a connected component in the background.

Now, how can we characterize the fact that some tunnels are created or removed in an object ? One could think of counting the number of tunnels using the Euler characteristic as well as the numbers of connected components (Conditions i and ii of Definition 10.1) are unchanged. The problem of such a characterization can be summed up by observing the case of the object X depicted in Figure 10.3 analyzed with the $6-$adjacency relation. In this case, the two objects depicted have the same number of connected components as their complements. Furthermore, they have the same Euler characteristic (equal to 0). However, the voxel x which has been removed from the object of Figure 10.3(a) to obtain the object the continuous analogue of which is depicted in Figure 10.3(c), is obviously not simple. The fact is that a tunnel was created and another one was removed by deletion of the voxel x, so that the Euler characteristic is left unchanged.



(a) To views of an object X and a voxel x.

(b) $C_6(X)$

(c) $C_6(X\setminus\{x\})$

Figure 10.3: A case when a tunnel is created whereas another one is simultaneously removed.

In fact, the problem is that the Euler characteristic does not allow to check the fact that tunnels have *moved*. However, it can be used to show that some tunnels are created or removed locally (i.e. in the neighborhood of a voxel) but not globally. Thus, the characterization which will be recalled in next section is correct since it characterizes locally the configurations for which a voxel can be removed, preserving the topological properties of the object. But this characterization is valid since it is based on the fact that "large" tunnels removal implies some local disconnections in the object.

10.1.3 Characterization of simple voxels using $\chi_n(N_{26}(x) \cap X)$

Using the Euler characteristic, Tsao and Fu in [101] have given the following local characterization for 3D simple voxels. In this definition, the Euler characteristic is used in order to check that the number of tunnels does not change in the neighborhood of the voxel considered.

Definition 10.2 ([101]) *Let X be a subset of $\mathbb{Z}^3$ and $(n, \overline{n}) \in \{(6, 26), (26, 6)\}$. The point x is an $n-$simple if and only if the three following conditions are satisfied :*

i) *x is $n-$adjacent to only one $n-$connected component of $N_{26}(x) \cap X$.*

ii) *x is $\overline{n}-$adjacent to only one $\overline{n}-$connected component of $N_{26}(x) \cap \overline{X}$.*

iii) $\chi_n(X \cap N_{26}(x)) = \chi_n(\{x\} \cup (X \cap N_{26}(x)))$.

Following this characterization, the point whose $26-$neighborhood is depicted in Figure 10.4 is not $6-$simple since Condition *iii*) of Definition 10.2 is not satisfied. Indeed, in this case, $C_6(\{x\} \cup (X \cap N_{26}(x)))$ has a tunnel whereas $C_6(X \cap N_{26}(x))$ does not. However, in the case of Figure 10.5, the point x is not $6-$simple since it is $6-$adjacent to two $6-$connected components of the object in its $26-$neighborhood. Nevertheless, in this case, Condition *iii*) of Definition 10.2 is satisfied but not Condition *i*). In this example, we see that removal of *large* tunnels is locally characterized by some disconnection in the neighborhood of a voxel. Indeed, the fact that a tunnel is removed by deletion of the voxel x of Figure 10.3(a) is detected since the voxel x is adjacent to two $6-$connected components of X in its $26-$neighborhood. Following this idea, G. Bertrand found some reduced neighborhoods for which *small* tunnels are also characterized by such localized disconnection : the geodesic neighborhoods.

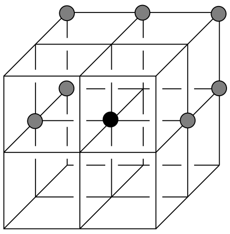


Figure 10.4: A *local* tunnel for $n =$ 6 or $n = 6+$.

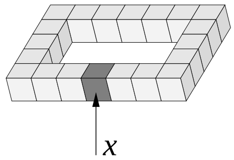


Figure 10.5: A *not local* tunnel.

10.2 Characterization using geodesic neighborhoods

In latter subsection, we saw that the Condition *iii*) of Definition 10.2 was useful in order to characterize the presence of small tunnels in the neighborhood of a voxel. However, for *larger* tunnels, it appears that local connectivity may be sufficient. In fact, it is possible to consider some smaller neighborhoods than the 26−neighborhood when checking the $n-$simplicity of a voxel. Then, connectivity considerations within such reduced neighborhoods are sufficient to check the possible deletion of small tunnels as well as larges tunnels, in the object and in its complement. Of course, global connectivity changes are also characterized by some considerations involving small neighborhoods. Then, the *geodesic neighborhoods* have been introduced by G. Bertrand in [8] and we recall here their definition. In the sequel of this section, X is a subset of $\mathbb{Z}^3$ and x is a voxel of X. First, for $n \in \{6, 6+, 18, 26\}$ we recursively define the sets $N_n^k(x, X)$ for $k \in \{1, 2, 3\}$ by :

– $N_n^1(x, X) = N_n(x) \cap X$, and

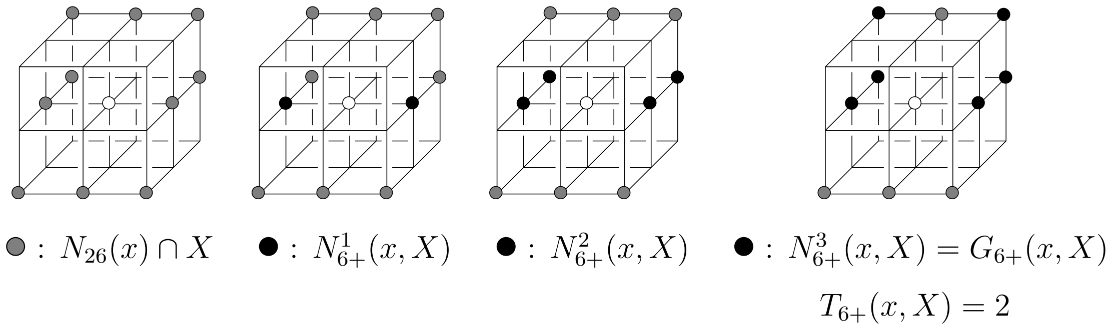
– $N_n^k(x, X) = N_n^{k-1}(x, X) \cup \{y \in N_{26}(x) \mid \exists z \in N_n^{k-1}(x, X),\ y \in N_n(z)\ \}$.

Definition 10.3 (geodesic neighborhood [8]) *The geodesic $n-$neighborhood of x in X denoted by $G_n(x, X)$ for $n \in \{6, 6+, 18, 26\}$ is defined as follows :*

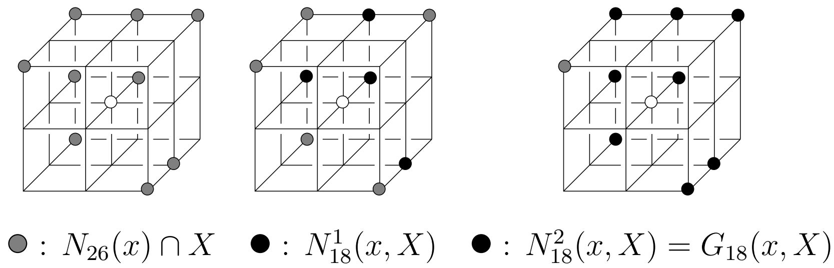
– $G_6(x, X) = N_6^2(x, X)$ – $G_{6+}(x, X) = N_6^3(x, X)$

– $G_{18}(x, X) = N_{18}^2(x, X)$ – $G_{26}(x, X) = N_{26}(x) \cap X$

The geodesic neighborhood $G_n(x, X)$ can be seen as the set obtained after a finite number of morphological dilatations of the point x inside $N_{26}(x) \cap X$ using the $n-$adjacency elementary ball as a structuring element, set from which the voxel x is removed. Some example of such sets are depicted in Figure 10.6.



(a) Construction of $G_{6+}(x, X)$



(b) Construction of $G_{18}(x, X)$

Figure 10.6: Examples of geodesic neighborhoods

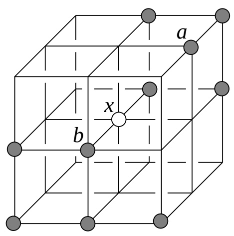
Definition 10.4 (topological numbers) *We define the* topological number *associated to x and X, and we denote by $T_n(x, X)$ the number of $n-$connected components of $G_n(x, X)$.*

Now, G. Bertrand stated and proved the following local characterization for simples points in [8] for $(n, \overline{n}) \in \{(6+, 18), (18, 6+)\}$ and together with G. Malandain in [11] for $(n, \overline{n}) \in \{(6, 26), (26, 6)\}$.

Proposition 10.1 *The point $x \in X \subset \mathbb{Z}^3$ is $n-$simple for X if and only if $T_n(x, X) = 1$ and $T_{\overline{n}}(x, X) = 1$.*

This proposition provides a local and efficient characterization of simple voxels in $\mathbb{Z}^3$ which does not require the computation of the Euler characteristic of the neighborhood of the voxel x, unlike Definition 10.2.

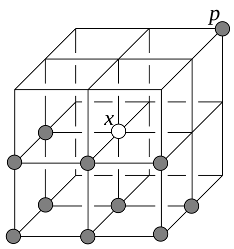
We have depicted in Figure 10.7 several local configurations of voxels which are either $n-$simple or not $n-$simple for $n \in \{6, 6+, 18, 26\}$. For example, the voxel x of Figure 10.7(a) is not $18-$simple since the removal of this voxel would either create a new $18-$connected component or remove a (large) tunnel of the object. However, the same



● : $N_{26}(x) \cap X$

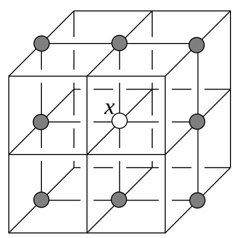
x is not 18−simple since $T_{18}(x, X) = 2$
x is 26−simple since $T_{26}(x, X) = T_6(x, \overline{X}) = 1$

(a)



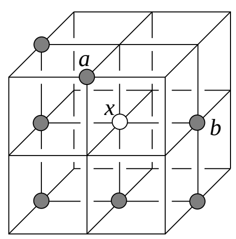
x is 18−simple since $T_{18}(x, X) = T_{6+}(x, \overline{X}) = 1$
x is not 26−simple since $T_{26}(x, X) = 2$

(b)



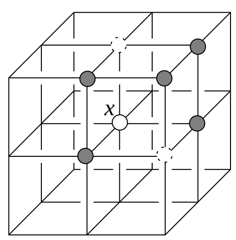
x is neither 26−simple nor 18−simple
since $T_{26}(x, \overline{X}) = T_{18}(x, \overline{X}) = 2$

(c)



x is 18−simple since $T_{18}(x, X) = T_{6+}(x, \overline{X}) = 1$
x is not 26−simple since $T_6(x, \overline{X}) = 2$

(d)



x is not 6−simple since $T_6(x, X) = 2$
x is (6+)−simple since $T_{6+}(x, X) = T_{18}(x, \overline{X}) = 1$

(e)

Figure 10.7: Simple and not simple voxels in $\mathbb{Z}^3$.

voxel is $26-$simple since the two points a and b of this figure are $26-$connected. In the case of Figure 10.7(b), the point x is not $26-$simple since the removal of x would either create a new $26-$connected component or remove a $26-$tunnel of X; whereas it is $18-$simple since the voxel p is not $18-$adjacent to x so that a similar disconnection or tunnel deletion cannot occur.

The case of Figure 10.7(d) need a more detailed description. Indeed, in this case, the point x is not $26-$simple since $T_6(x, \overline{X}) = 2$ whereas it is $18-$simple since $T_{18}(x, X) = T_{6+}(x, \overline{X}) = 1$. In other words, this point is not $26-$simple since its removal would either merge two $6-$connected components of $\overline{X}$ or create a (large) $6-$tunnel in $\overline{X}$. In Chapter 12, we will prove that in this case, either a $26-$tunnel of X is created or two $6-$connected components of $\overline{X}$ are merged by deletion of x.

In the case of Figure 10.7(e) we have illustrated the usefulness of the geodesic neighborhoods in order to avoid the use of the Euler characteristic for the detection of small tunnels. Indeed, the set of grey points in this figure together with x constitutes a tunnel in X for $(n, \overline{n}) = (6, 26)$ (the two dotted points of $\overline{X}$ are $26-$adjacent) and we have $T_6(x, X) = 2$. Nevertheless, for $(n, \overline{n}) = (6+, 18)$ we observe that these voxels do not constitute a tunnel (the points of $\overline{X}$ depicted with dotted lines are not $18-$adjacent) and we have $T_{6+}(x, X) = 1$.

Now, we can find two limits to the use of the Euler characteristic in order to characterize topology preservation in $\mathbb{Z}^3$. First, it is very difficult to use the Euler characteristic to formalize the fact that the tunnels of Figures 10.3(b) and 10.3(c) are *distinct*. And this limit makes the characterization ambiguous. On a second hand, another definition for tunnels has been used by Morgenthaler in [75] and also by Bertrand in [8, 11]. This definition was introduced in order to avoid the latter drawback of the Euler characteristic, and says that a tunnel is detected whenever there exists a closed path in an object which is not reducible in the object, following the idea of the definition of *simply connected subsets of a topological space* which was recalled in Part I. This idea, which motivated the introduction by Kong of the digital fundamental group ([45]), was then used to formalize in proofs the sentences “no tunnel of X is created or removed” and “no tunnel of $\overline{X}$ is created or removed” of Definition 10.1. However, some arguments are no longer valid in this new context when they use both the two possible definitions for tunnels. An example of such an argument is summarized as follows : Morgenthaler stated in [75] that $NH(X) =$

$NH(\overline{X})$ where $NH(X)$ is the number of tunnels in X (*Holes* using the terminology of [75]). This property is deduced from the definition of the Euler characteristic. Now, if a tunnel of the complement of X is created by deletion of the point x, i.e. there exists a closed path π in $\overline{X} \cup \{x\}$ which is not homotopic to a trivial path in $\overline{X} \cup \{x\}$ but no closed path in $\overline{X}$ is homotopic to the path π ; then since $NH(\overline{X}) = NH(X)$ we deduce that a tunnel has been created in X.

In practice, it is not so difficult to prove that a tunnel is created in the complement of an object by deletion of a voxel (see Section 12.1), whereas it is difficult to prove the same thing for the object without using the previous argument. But the fact is that this latter argument uses some close but different approaches to the definition of a tunnel which involves two distinct topological invariants : the Euler characteristic and an informal version of the digital fundamental group. Nevertheless, the link between these two invariants has not been properly proved in the digital context, and this can appear as unsatisfactory. Now, a convenient formalization was proposed by Kong using the digital fundamental group, which finalizes the ideas given by Morgenthaler in [75] and will be presented in the next subsection.

10.3 Using the Digital Fundamental Group

The digital fundamental group, introduced by Kong in [45] and the definition of which is recalled in Section 2.3, allows to formalize the characterization of simple voxels as follows, which is unambiguous unlike Definition 10.1 :

Definition 10.5 *Let $X \subset \mathbb{Z}^3$ and $x \in X$. The voxel x is said to be $n-$simple for X if the four following conditions are satisfied :*

i) X and $X \setminus \{x\}$ have the same number of $n-$connected components.

ii) $\overline{X}$ and $\overline{X} \cup \{x\}$ have the same number of $\overline{n}-$connected components.

iii) For each voxel B in $X \setminus \{x\}$, the group morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i : X \setminus \{x\} \longrightarrow X$ is an isomorphism.

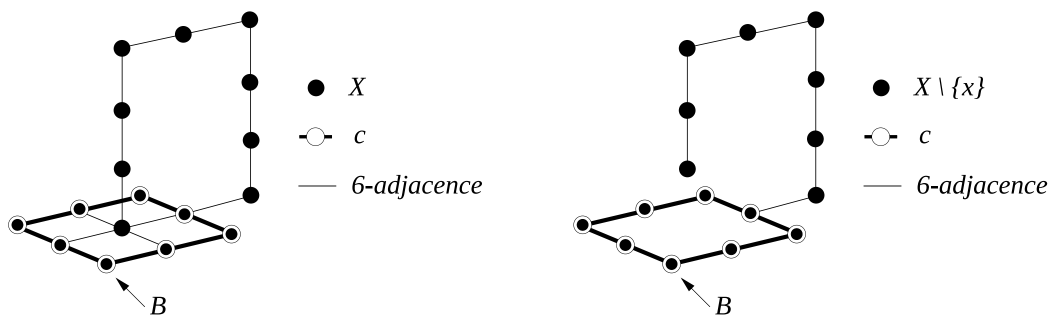
iv) For each point B' in $\overline{X}$, the group morphism $i'_* : \Pi_1^{\overline{n}}(\overline{X}, B') \longrightarrow \Pi_1^{\overline{n}}(\overline{X} \cup \{x\}, B')$ induced by the inclusion map $i' : \overline{X} \longrightarrow \overline{X} \cup \{x\}$ is an isomorphism.

The definition of the morphism induced by an inclusion map can be found in Section 2.3.2.

This definition appears as very convenient one since it allows to formalize the fact that X and $X \setminus \{x\}$ [resp. $\overline{X}$ and $\overline{X} \cup \{x\}$] have not only the same number of tunnels, but also the fact that these tunnels are the same. As an illustration of this property, we shall come back to the example given Page 156 (Figure 10.3).

We have depicted in Figure 10.8(a) an object X (black points) and a closed path c from a point B to B in X. Then, the homotopy class $[c]_{\Pi_1^6(X,B)}$ is obviously the identity element of $(\Pi_1^6(X,B),*)$ since c is $6-$homotopic to the path (B,B) in X. However, the same path c is obviously not $6-$reducible in $X \setminus \{x\}$ (Figure 10.8(b)). It follows that the group morphism i_* from $\Pi_1^6(X \setminus \{x\}, B)$ to $\Pi_1^6(X,B)$ induced by the inclusion of $X \setminus \{x\}$ in X is not one to one. Indeed, $i_*([c]_{\Pi_1^6(X\setminus\{x\},B)}) = i_*([(B,B)]_{\Pi_1^6(X\setminus\{x\},B)})$ whereas $[c]_{\Pi_1^6(X\setminus\{x\},B)} \neq [(B,B)]_{\Pi_1^6(X\setminus\{x\},B)}$.

On a second hand, the equivalence class of the path c' in the object depicted in Figure 10.9(a), which is obviously not equal to the identity element of $(\Pi_1^6(X,B),*)$, cannot by reached by the morphism i_* since no closed path of $X \setminus \{x\}$ is $6-$homotopic to the path c' in X. In other word, the morphism i_* is not onto.



(a) An object X.

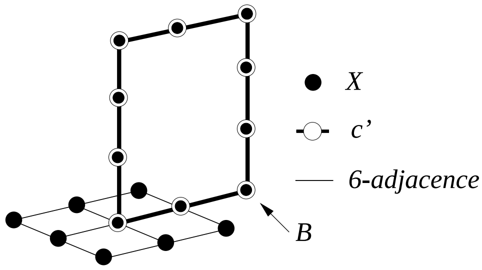
(b) The object $X \setminus \{x\}$.

Figure 10.8: The morphism induced by the inclusion of $X \setminus \{x\}$ in X is not one to one.

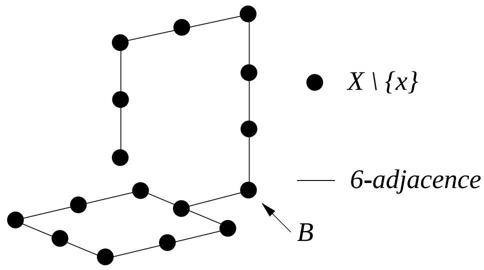
Finally, we see that Condition *iii)* of Definition 10.5 seems sufficient to detect when, simultaneously, a hole is created and a hole is removed.

10.4 Other approaches for a local characterization

First, we have to mention here some works the purpose of which is to find some efficient local characterizations of simple voxels. For example, Saha et al. give in [96] a local



(a) An object X.



(b) The object $X \setminus \{x\}$.

Figure 10.9: The morphism induced by the inclusion of $X \setminus \{x\}$ in X is not onto.

characterization of simple voxels for $(n, \overline{n}) = (26, 6)$ (equivalent to the one given in [11]) which leads to an efficient algorithm to detect this kind of points.

Moreover, some other approaches can be used to characterize simple voxels. Indeed, in [46], T.Y. Kong gives a local characterization of 3D simple points, based an attachment sets, which has the good property of being easily visualized.

Chapter 11

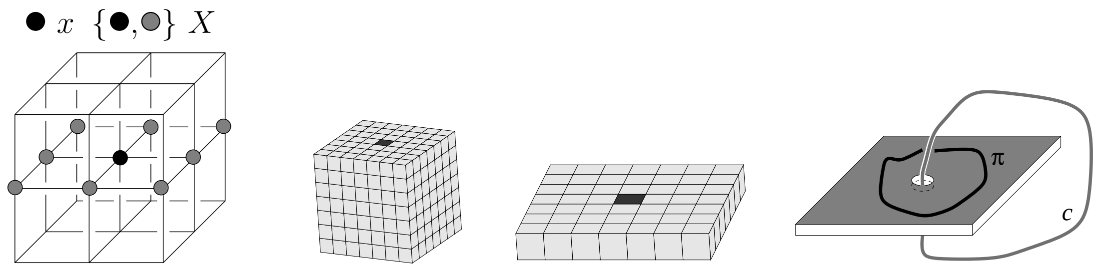
The linking number

In the previous chapter, we have recalled the definition of simple voxels in $\mathbb{Z}^3$ and given few formal definitions of this property. In this part, we are interested by the definition of a global characterization of simple voxels which involves the digital fundamental but which is more concise than Definition 10.5. Indeed, our purpose is to state that the Condition *iv*) of Definition 10.5 is implied by Conditions *i*), *ii*) and *iii*) of this definition. This idea will be illustrated in the next section and constitues one of the motivations of the definition of the digital linking number.

11.1 Motivation

Condition *iii*) of Definition 10.5 states that a voxel is simple if any tunnel of an object still exists and *at the same place* in the object from which the voxel has been removed. On the other hand it also requires that no new tunnel is created in the object by deletion of the voxel. Let us consider the local configuration depicted in Figure 11.1(a) where the central voxel x is not $n-$simple for $n \in \{6, 6+, 18, 26\}$. Indeed, this configuration may be a part of the hollow cube X' depicted in Figure 11.1(b) so that two background components are merged by removal of this voxel (Condition *ii* of Definition 10.5). Nevertheless, this local configuration may also be a part of the object X'' of Figure 11.1(c) so that a tunnel appears in $\overline{X''} \cup \{x\}$ as illustrated by Figure 11.1(d). Indeed, the closed path c of Figure 11.1(d) is not reducible in $\overline{X''} \cup \{x\}$ so that $\overline{X''} \cup \{x\}$ has a tunnel whereas $\overline{X''}$ does not. . We will see in Section 12.1 how the existence of such a tunnel can be proved in this case. Now, we also observe in Figure 11.1(d) that the black path π in $X'' \setminus \{x\}$

is also not reducible in $X'' \setminus \{x\}$. However, this very intuitive fact is not easy to prove using the tools which are available in the digital framework. Now, a way to achieve such a proof is to state that the two paths π and c of Figure 11.1(d) are *linked* and cannot be unlinked if the only allowed deformation is an homotopic deformation of the paths π and c respectively in $X \setminus \{x\}$ and in $\overline{X} \cup \{x\}$ in the complement of each other.

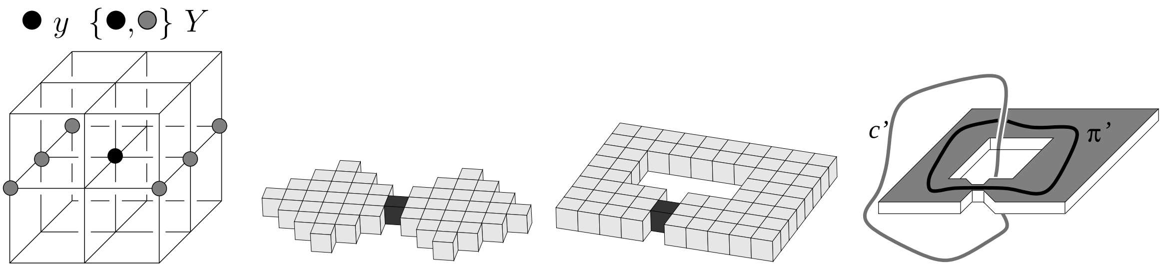


(a) x and $N_{26}(x) \cap X$ (b) X' (c) X'' (d) $X'' \setminus \{x\}$

Figure 11.1: A tunnel is created in $\overline{X}$ if and only if one is created in X (see the proof of Theorem 16).

Similarly, let us observe the configuration of Figure 11.2(a). If such a configuration appears as in the object Y' of Figure 11.2(b), then the point y is not simple since a connected component of Y' is sliced is two pieces by removal of the voxel y. Now, if the same configuration appears in the object Y'' of Figure 11.2(c), then we may prove easily (and in a similar way to the case of the path c in Figure 11.1(c)) that the closed path π' is not reducible in Y'' so that Y'' has a tunnel whereas $Y'' \setminus \{y\}$ obviously does not. Again, using the linking property previously mentioned, we should also be able to prove that the closed path c' of Figure 11.2(d) is also not reducible in $\overline{Y''}$ so that a (the) tunnel of $\overline{Y''}$ will also have disappeared after deletion of the voxel y.

Then, this gives a first idea of the fact that the creation [resp. the deletion] of a tunnel in an object by removal of a point is strictly related to the creation [resp. deletion] of a tunnel in its complement. Now, such a property is proved immediately when considering only the number of tunnels as counted using the Euler characteristic. Indeed, following [75], if we denote by $NCC(X)$, $NC(X)$ and $NT(X)$ respectively the number of $n-$connected components, cavities and tunnels of a subset X of $\mathbb{Z}^3$, then it is straightforward that $NCC(X) = NC(\overline{X})$ and $NC(X) = 1 + NCC(\overline{X})$. Now, the Euler characteristic of a digital object, following equation 10.1, may be computed by counting the number of



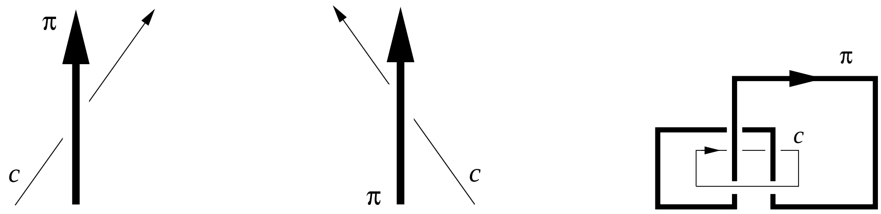
(a) y and $N_{26}(x) \cap Y$ (b) Y' (c) Y'' (d) Y''

Figure 11.2: A tunnel is removed in X if and only if one is removed in $\overline{X}$.

occurrences of a finite number of small patterns in the object as proposed in [75] and [74]. Then, following this computation method, [74] contains a proof of the fact that $\chi_{\overline{n}}(\overline{X}) = 1 + \chi_n(X)$. Finally, using this latter equality and the former ones, it comes that the number of tunnels in X is equal to the number of tunnels in $\overline{X}$. However, as explained before, the number of tunnels is a less precise invariant than the fundamental group. Nevertheless, the linking number will allow to establish such a connection using the only topological invariant considered in Definition 10.5 : the digital fundamental group.

11.2 The digital linking number

In this section, we define the linking number between two closed paths of voxels which do no intersect one each other. This number is nothing but the linking number of the continuous analogue of the two digital curves as defined in knot theory. This linking number counts the number of times a given closed path is interlaced around another one. Since our further goal is to apply this tool to prove theorems about topology in a digital space, we are interested by the linking number between a closed $n-$path and a closed $\overline{n}-$path where $(n, \overline{n}) \in \{(6+, 18), (6, 26), (18, 6+), (26, 6)\}$. We give three examples of pairs of closed paths and their associated linking numbers in Figure 11.2. Classically, the linking number is computed by algebraically counting the occurrences of crosses like those depicted in Figure 11.3 in a 2 dimensional regular projection of the paths (see [85]). In our case, we define the linking number in such a way that it can be immediately obtained by integer only computations using the coordinates of the voxels constituting the paths.

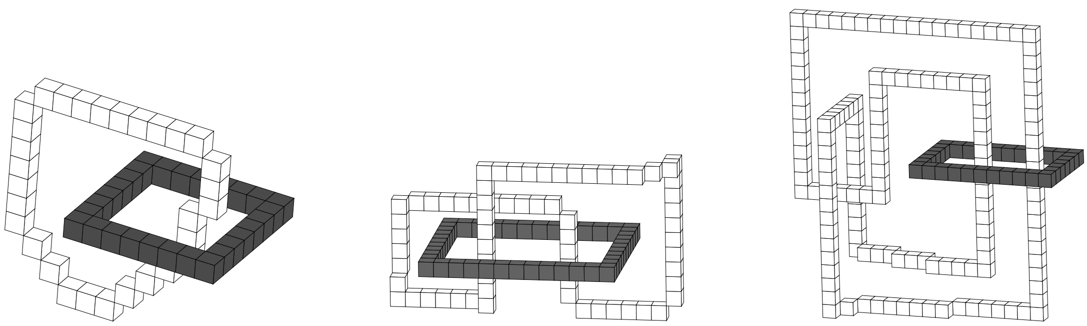


(a) Count -1 for each occurrence of such a cross in the projection.

(b) Count +1 for each cross of this type.

(c) The linking number associated to this projection is -2.

Figure 11.3: The way to compute the "classical" linking number from a regular projection of two closed paths c and π.



(a) A closed 18−path and a closed 6−path with a linking number of ± 1.

(b) A closed 18−path and a closed 6−path with a linking number of ± 2.

(c) The Whitehead's link, the linking number of which is 0.

Figure 11.4: Three kinds of links.

For example, the linking number can be computed immediately for paths as depicted by Figure 11.5. We give the basic idea of the computation in this case. First, we choose to compute the linking number using a "pseudo-2D" projection of the two paths on the plane which contains the first two coordinates axes. We call this projection a "pseudo-2D" one since the data of the third coordinate of the voxels is of course never forgotten. Then we observe that the only voxel of the grey path which has a common projection with some white voxels (exactly four ones) is the point x_i. Then, we look for voxels of the white path which have a greater third coordinate than x_i and the same projection as x_i (as the voxels a and b in Figure 11.5). For each such voxel, a contribution depending on the position of the next and previous voxel of the white path which have a distinct projection from x_i is computed. In this example, the two contributions of the voxels a and b will have opposite signs. The sum of these contributions is the linking number between the two paths, zero in this case. In fact, using half contributions exactly as in the case of the intersection number introduced in previous part, we will count the number of transversal oriented intersections between the projection of the two paths; furthermore only when the first one goes before the second one (according to the third coordinate axe). Thus, in Figure 11.7(c) and Figure 11.7(d) we have marked in dark grey the two sets of consecutive voxels of the first 6−path of Figure 11.7(a) which have a common projection with some voxels (in light grey) of the 18−path. Then, taking into account the position of the voxels marked with crosses allows to check if such a *projective intersection* is transversal (an example of a tangent intersection is given in Figure 11.8). Finally, counting algebraically the three projective transversal intersections in Figure 11.7 leads to a linking number of ± 1 depending on the parameterization of the paths.

Notation 11.1 We will denote by $\mathcal{P}$ the following map :

$$\begin{array}{rccc} \mathcal{P} : & \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \\ & (x^1, x^2, x^3) & \longmapsto & (x^1, x^2) \end{array}$$

Now, we need to define the *predecessor* and the *successor* of a voxel x_i of an $n-$path c according to a projection $\mathcal{P}$ which are the first voxels which come respectively before and after x_i in the parameterization of c and whose projection by $\mathcal{P}$ is distinct from the projection of x_i. Observe that the two number *Pred* and *Succ* defined here are subscripts of the parameterization of the paths. Finally, the data of these subscripts will allow to define an orientation for intersections in the projection of two 3d closed paths.

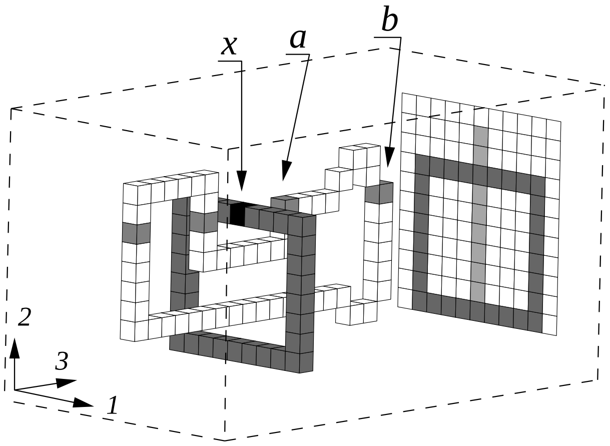


Figure 11.5: Two 3D closed paths and their projection, their linking number is 0.

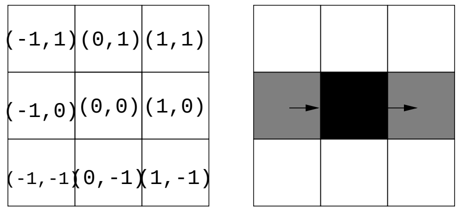
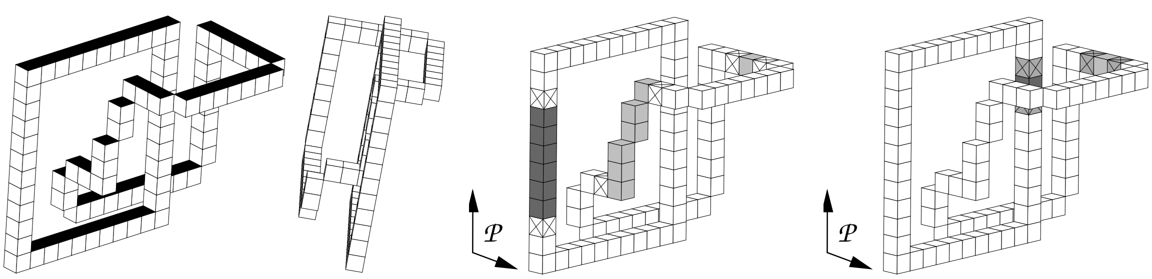
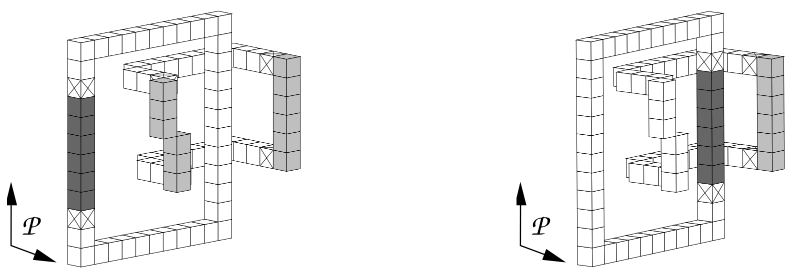


Figure 11.6: A projective movement.



(a) (b) (c)

Figure 11.7: Example of projective transversal intersections between two closed paths in $\mathbb{Z}^3$.



(a) (b)

Figure 11.8: Example of projective tangent intersections between two closed paths in $\mathbb{Z}^3$.

Definition 11.1 (*Pred* and *Succ*) *Let $c = (x_i)_{i=0,\dots,q}$ be a closed $n-$path and x_i be a voxel of c for $i \in [0, q]$. Then, $Succ_c(i)$ is the lowest integer l greater than i such that $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$; if such an integer l does not exist then $Succ_c(i)$ is the lowest $l < i$ such that $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$. If in turn such an l does not exist then, clearly $\mathcal{P}(x_i) = \mathcal{P}(x_l)$ for all $l \in \{0, \dots, q\}$ and we define $Succ_c(i) = i$.*

Similarly, $Pred_c(i)$ is the preceding subscript l of i in the cyclic parameterization of c such that $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$, or $Pred_c(i) = i$ if $\mathcal{P}(x_i) = \mathcal{P}(x_l)$ for all $l \in \{0, \dots, q\}$.

Still in order to count oriented transversal intersections in the projection of the two paths, we define the following notion of a projective movement.

Definition 11.2 (projective movement) *Let $c = (x_i)_{i=0,\dots,q}$ be a closed $n-$path and $i \in \{0, \dots, q\}$. Let V be the $8-$neighborhood of $(0, 0)$ in the plane, i.e. $V = (\{-1, 0, 1\} \times \{-1, 0, 1\}) \setminus \{(0, 0)\}$. We define the* projective movement *$P_c(i) \in V \times V$ associated to the subscript i of c by :*

$$P_c(i) = ((x^1_{Pred_\pi(i)} - x^1_i, x^2_{Pred_\pi(i)} - x^2_i), (x^1_{Succ_\pi(i)} - x^1_i, x^2_{Succ_\pi(i)} - x^2_i)) = (P_c(i)^{Pred}, P_c(i)^{Succ}).$$

The projective movement represents the position of the previous and the following voxels of x_i in c whose projection do not coincide with the projection of x_i. These positions are normalized in a 3×3 grid centered at the point $(0, 0)$ which is associated to the projection of x_i. Hence, the projective movement of the voxel x_i of Figure 11.5 is $((-1, 0), (1, 0))$ and can be seen as depicted by Figure 11.6. Note that this projective movement will be used only for subscripts $i \in \{0, \dots, q - 1\}$ such that $Pred_c(i) = i - 1$. Indeed, we have to count a single contribution to the linking number for any sequence of voxels of a path which have the same projection, and we arbitrary choose to count contributions at the first subscript of each such sequence.

Definition 11.3 (left and right) *Let $c = (x_i)_{i=0,\dots,q}$ be an $n-$path and V be the set introduced in Definition 11.2. One can parameterize the points of V using the counterclockwise order around the point $(0, 0)$. Then, given a projective movement $\mathcal{P} = P_c(i)$, we define the two sets $Left(\mathcal{P})$ and $Right(\mathcal{P})$ as follows :*

$Right(\mathcal{P})$ is the set of points met when looking after points of V from $\mathcal{P}^{Pred}$ to $\mathcal{P}^{Succ}$ following the counterclockwise order on V, excluding $\mathcal{P}^{Succ}$ and $\mathcal{P}^{Pred}$.

$Left(\mathcal{P})$ is the set of points met when looking after points of V from $\mathcal{P}^{Succ}$ to $\mathcal{P}^{Pred}$ following the counterclockwise order on V, excluding $\mathcal{P}^{Succ}$ and $\mathcal{P}^{Pred}$.

Example : *If* $\mathcal{P} = ((-1,0),(1,-1))$ *then* $Right(\mathcal{P}) = \{(-1,-1),(0,-1)\}$ *and* $Left(\mathcal{P}) = \{(1,0),(1,1),(0,1),(-1,1)\}$.

Notation 11.2 In the following we say that two paths π and c satisfy the property $\mathcal{H}(\pi,c)$ if π is a closed $n-$path for $n \in \{6,6+\}$ and c is closed $\overline{n}-$path such that $c^* \cap \pi^* = \emptyset$.

In the sequel of this part we consider $n \in \{6,6+\}$. Furthermore, and in order to shorten notations, we will use the following notation for intervals of integers.

Notation 11.3 Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with $a \leq b$. Then we use the following notations for intervals of integers :

$$\begin{aligned}
[a,b] &= \{i \in \mathbb{Z} \mid a \leq i \leq b\} \\
[a,b[&= \{i \in \mathbb{Z} \mid a \leq i \text{ and } i < b\} \\
]a,b] &= \{i \in \mathbb{Z} \mid a < i \text{ and } i \leq b\} \\
]a,b[&= \{i \in \mathbb{Z} \mid a < i \text{ and } i < b\}
\end{aligned}$$

Definition 11.4 (contribution to the linking number) *Let* $\pi = (y_k)_{k=0,\ldots,p}$ *and* $c = (x_i)_{i=0,\ldots,q}$ *be two closed paths such that* $\mathcal{H}(\pi,c)$ *holds. We define as follows* $W_{\pi,c}(k,i)$, *the* direct contribution to the linking number *of a couple* (k,i), *where* $0 \leq k \leq p$ *and* $0 \leq i \leq q$.

- *If* $y_k^3 > x_i^3$ *or* $\mathcal{P}(y_k) \neq \mathcal{P}(x_i)$ *or* $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ *or* $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ *then* $W_{\pi,c}(k,i) = 0$,
- *otherwise, let* $\mathcal{P}_\pi = P_\pi(k)$ *and* $\mathcal{P}_c = P_c(i)$ *be the projective movements associated to the subscripts* i *and* k *(note that in this case* $Pred_\pi(k) = k-1$ *and* $Pred_c(i) = i-1$*) :*
  - *If* $\mathcal{P}_\pi^{Pred} = \mathcal{P}_\pi^{Succ}$ *then* $W_{\pi,c}(k,i) = 0$,
  - *otherwise* $W_{\pi,c}(k,i) = W^-_{\pi,c}(k,i) + W^+_{\pi,c}(k,i)$ *where*

    $W^-_{\pi,c}(k,i) = -0.5$ *if* $\mathcal{P}_c^{Pred} \in Left(\mathcal{P}_\pi)$, $W^+_{\pi,c}(k,i) = -0.5$ *if* $\mathcal{P}_c^{Succ} \in Right(\mathcal{P}_\pi)$,

    $W^-_{\pi,c}(k,i) = 0.5$ *if* $\mathcal{P}_c^{Pred} \in Right(\mathcal{P}_\pi)$, $W^+_{\pi,c}(k,i) = 0.5$ *if* $\mathcal{P}_c^{Succ} \in Left(\mathcal{P}_\pi)$,

    $W^-_{\pi,c}(k,i) = 0$ *otherwise.* $W^+_{\pi,c}(k,i) = 0$ *otherwise.*

Definition 11.5 (linking number) *Let* $\pi = (y_k)_{k=0,\ldots,p}$ *and* $c = (x_i)_{i=0,\ldots,q}$ *be two closed paths such that* $\mathcal{H}(\pi,c)$ *holds. We define the* linking number of π and c *(denoted by* $L_{\pi,c}$*) by :*

$$L_{\pi,c} = \sum_{k=0}^{p-1} \sum_{i=0}^{q-1} W_{\pi,c}(k,i) \tag{11.1}$$

Notation 11.4 Given two closed paths $\pi = (y_k)_{k=0,\ldots,p}$ and $c = (x_i)_{i=0,\ldots,q}$, we denote :
For $i \in [0, q]$, $L^{\pi}_{\pi,c}(i) = \sum_{k=0}^{p-1} W_{\pi,c}(k, i)$ and for $k \in [0, p]$, $L^{c}_{\pi,c}(k) = \sum_{i=0}^{q-1} W_{\pi,c}(k, i)$.

11.3 A new topological invariant

Now, we state the two main results which are proved in this chapter about the invariance of the linking number up to an homotopic deformation of any the two paths for which it is defined. These very intuitive results, again very close to the similar result of the continuous case are proved in this chapter by using technical but very simple considerations about integer coordinates of voxels.

Theorem 14 *Let π and π' be two closed $n-$paths ($n \in \{6, 6+\}$) and c be a closed $\overline{n}-$path of $\mathbb{Z}^3$ such that $\pi^* \cap c^* = \emptyset$ and $\pi'^* \cap c^* = \emptyset$. If π is $n-$homotopic to π' in $\mathbb{Z}^3 \setminus c^*$ then $L_{\pi,c} = L_{\pi',c}$.*

Theorem 15 *Let π be a closed $n-$path ($n \in \{6, 6+\}$), let c and c' be two closed $\overline{n}-$paths of $\mathbb{Z}^3$ such that $\pi^* \cap c^* = \emptyset$ and $\pi^* \cap c'^* = \emptyset$. If c is $\overline{n}-$homotopic to c' in $\mathbb{Z}^3 \setminus \pi^*$ then $L_{\pi,c'} = L_{\pi,c'}$.*

As an illustration, one can be convinced that any $18-$homotopic deformation of the $18-$closed white path of Figure 11.4(b) in the complement of the $6-$closed grey path cannot change the linking number associated to the two paths.

11.4 Useful properties

In this section, we give the definition of the indirect contribution to the linking number which allows to compute the linking number by looking after voxels of the 18 or $26-$path and counting the crossings with the projection of the $6-$path. This leads to an equivalent definition of the linking number, which allows to prove in a very similar way two propositions about an additive property of the linking number by concatenation of the paths (Proposition 11.3 and Proposition 11.2 of this section). This additive property will be used in the next section to prove the two main theorems of this chapter.

11.4.1 An equivalent definition of the linking number

Definition 11.6 (indirect contribution to the linking number)
Let $\pi = (y_k)_{k=0,\ldots,p}$ and $c = (x_i)_{i=0,\ldots,q}$ be two closed paths such that $\mathcal{H}(\pi, c)$ holds. We define as follows $M_{c,\pi}(i,k)$, the indirect contribution to the linking number *of a couple (i,k) where $0 \le i \le q$ and $0 \le k \le p$.*

- *If $y_k^3 > x_i^3$ or $\mathcal{P}(y_k) \neq \mathcal{P}(x_i)$ or $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ or $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ then $M_{c,\pi}(i,k) = 0$,*
- *otherwise, let $\mathcal{P}_\pi = P_\pi(k)$ and $\mathcal{P}_c = P_c(i)$ be the projective movements associated to the subscripts i and k:*

  — *If $\mathcal{P}_c^{Pred} = \mathcal{P}_c^{Succ}$ then $M_{c,\pi}(i,k) = 0$,*

  — *otherwise $M_{c,\pi}(i,k) = M^-_{c,\pi}(i,k) + M^+_{c,\pi}(i,k)$ where*

$M^-_{c,\pi}(i,k) = +0.5$ *if* $\mathcal{P}_\pi^{Pred} \in Left(\mathcal{P}_c)$,	$M^+_{c,\pi}(i,k) = +0.5$ *if* $\mathcal{P}_\pi^{Succ} \in Right(\mathcal{P}_c)$,
$M^-_{c,\pi}(i,k) = -0.5$ *if* $\mathcal{P}_\pi^{Pred} \in Right(\mathcal{P}_c)$,	$M^+_{c,\pi}(i,k) = -0.5$ *if* $\mathcal{P}_\pi^{Succ} \in Left(\mathcal{P}_c)$,
$M^-_{c,\pi}(i,k) = 0$ *otherwise.*	$M^+_{c,\pi}(i,k) = 0$ *otherwise.*

Lemma 11.1 *Let $\pi = (y_k)_{k=0,\ldots,p}$ and $c = (x_i)_{i=0,\ldots,q}$ be two closed paths such that $\mathcal{H}(\pi, c)$ holds. Then $W_{\pi,c}(k,i) = M_{c,\pi}(i,k)$.*

Proof : The first condition of Definition 11.4 and the one of Definition obn11.6 are identical. Now, if $\mathcal{P}_\pi^{Pred} = \mathcal{P}_\pi^{Succ}$ then $W_{\pi,c}(k,i) = 0$ but it is then clear that whatever be the configuration $\mathcal{P}_c$, we have $M_{c,\pi}(i,k) = 0$ since $\mathcal{P}_\pi^{Pred}$ and $\mathcal{P}_\pi^{Succ}$ will either both belong to the same side of $\mathcal{P}_c$ or be equal to $\mathcal{P}_c^{Pred}$ or $\mathcal{P}_c^{Succ}$. Similarly, if $\mathcal{P}_c^{Pred} = \mathcal{P}_c^{Succ}$ then $W_{\pi,c}(k,i) = M_{c,\pi}(i,k) = 0$.

If $\mathcal{P}_\pi^{Pred} \neq \mathcal{P}_\pi^{Succ}$ and $\mathcal{P}_c^{Pred} \neq \mathcal{P}_c^{Succ}$, we should evaluate $W_{c,\pi}(i,k)$ depending on the positions of the points $\mathcal{P}_c^{Pred}$ and $\mathcal{P}_c^{Succ}$, which immediately gives the positions of $\mathcal{P}_\pi^{Pred}$ and $\mathcal{P}_\pi^{Succ}$ relative to $\mathcal{P}_c$. In all case we only have to observe that $M_{c,\pi}(i,k) = W_{\pi,c}(k,i)$. Figure 11.9 gives an overview of the fourteen configurations of projective movements which can occur between an $n-$path and an $\overline{n}-$path. The reader can check that the direct and indirect contributions of the intersection point are equal. □

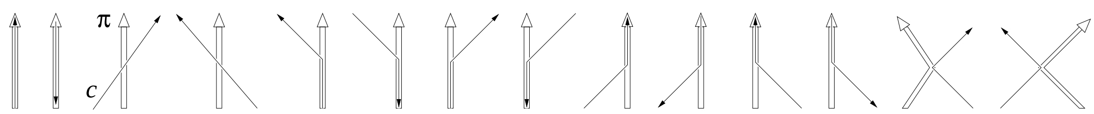


Figure 11.9: The 14 possible crossing ways in a projective movement.

Remark 11.1 *From Lemma 11.1 we have :* $L_{\pi,c} = \sum_{i=0}^{q-1}\sum_{k=0}^{p-1} M_{c,\pi}(i,k)$. *Furthermore, it is clear that the linking number is not dependent to the choice of a parameterization for any of the two paths as soon as the orientation of the considered path is preserved.*

11.4.2 The concatenation property

Proposition 11.2 *Let* π_1, π_2 *be two closed* $n-$*paths with the same extremities and* c *be a closed* $\overline{n}-$*path such that* $\mathcal{H}(c,\pi_1)$ *and* $\mathcal{H}(c,\pi_2)$ *hold. Then,* $L_{\pi_1.\pi_2,c} = L_{\pi_1,c} + L_{\pi_2,c}$.

Proof : Let $c = (x_0,\dots,x_q)$, $\pi_1 = (z_0,\dots,z_{p_1})$, $\pi_2 = (t_0,\dots,t_{p_2})$ and $\pi_1.\pi_2 = (y_0,\dots,y_{p_1+p_2})$.

From Definition 11.5, we have to prove that $\sum_{i=0}^{q-1} L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) = \sum_{i=0}^{q-1} L^{\pi_1}_{\pi_1,c}(i) + \sum_{i=0}^{q-1} L^{\pi_2}_{\pi_2,c}(i)$.

More precisely, it is sufficient to prove that $L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) = L^{\pi_1}_{\pi_1,c}(i) + L^{\pi_2}_{\pi_2,c}(i)$ for any $i \in [0,q]$. From Definition 11.4, both terms of the previous equality are equal to zero if $\mathcal{P}(x_{i-1}) = \mathcal{P}(x_i)$ or if $\mathcal{P}(x_{i-1}) = \mathcal{P}(x_{Succ_c(i)})$. Therefore, we have to investigate the case when the projective movement $\mathcal{P} = P_c(i)$ (see Definition 11.4) is not trivial in the sense that $\mathcal{P}^{Pred} \neq \mathcal{P}^{Succ}$.

In this case, we prove that : $\sum_{k=0}^{p_1+p_2-1} M_{c,\pi_1.\pi_2}(i,k) = \sum_{k=0}^{p_1-1} M_{c,\pi_1}(i,k) + \sum_{k=0}^{p_2-1} M_{c,\pi_2}(i,k)$.

• If both π_1 and π_2 are closed paths the projection of which is reduced to a single point, i.e. $\mathcal{P}(z_0) = \mathcal{P}(z_k) = \mathcal{P}(t_0)$ for any $k \in [0,p_1-1]$ and $\mathcal{P}(t_0) = \mathcal{P}(t_k)$ for any $k \in [0,p_2-1]$ then it is immediate that $L_{\pi_1,c} = L_{\pi_2,c} = L_{\pi_1.\pi_2,c} = 0$.

• If the path π_2 has a projection reduced to a single point (i.e. $Succ_{\pi_2}(0) = 0$) and π_1 has a projection which is not reduced to a single point (i.e. $Succ_{\pi_1}(0) \neq 0$) then $L_{\pi_2,c} = 0$ and we prove that $L_{\pi_1,c} = L_{\pi_1.\pi_2,c}$. Indeed, in this case and for any $i \in [0,q-1]$:

$$\begin{aligned} L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) &= \sum_{k=0}^{Succ_{\pi_1.\pi_2}(0)-1} M_{c,\pi_1.\pi_2}(i,k) + \sum_{k=Succ_{\pi_1.\pi_2}(0)}^{Pred_{\pi_1.\pi_2}(0)-1} M_{c,\pi_1.\pi_2}(i,k) \\ &+ \sum_{k=Pred_{\pi_1.\pi_2}(0)}^{p_1+p_2-1} M_{c,\pi_1.\pi_2}(i,k) \end{aligned}$$

But from the definition of $Succ_{\pi_1.\pi_2}(0)$ and $Pred_{\pi_1.\pi_2}(0)$ and due to the fact that the indirect contribution of a couple (k,i) is equal to 0 in the case when $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ we obtain that :

$$\sum_{k=Pred_{\pi_1.\pi_2}(0)}^{p_1+p_2-1} M_{c,\pi_1.\pi_2}(i,k) + \sum_{k=0}^{Succ_{\pi_1.\pi_2}(0)-1} M_{c,\pi_1.\pi_2}(i,k) = M_{c,\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1,k)$$

We also observe that : $Pred_{\pi_1.\pi_2}(0) = Pred_{\pi_1}(0) \in]0, p_1[$ since $\mathcal{P}(y_k) = \mathcal{P}(y_{p_1})$ for $k \in [p_1, p_1 + p_2]$. But $y_j = z_j$ for all $j \in [0, \dots, p_1]$ so that $y_{Pred_{\pi_1.\pi_2}(0)} = z_{Pred_{\pi_1}(0)}$ and $y_{Pred_{\pi_1.\pi_2}(0)+1} = z_{Pred_{\pi_1}(0)+1}$. On the other hand, $Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1) = Succ_{\pi_1.\pi_2}(0)$ from the definition of *Succ* and *Pred*. But $Succ_{\pi_1.\pi_2}(0) \in]0, p_1[$, so that $Succ_{\pi_1.\pi_2}(0) = Succ_{\pi_1}(0) = Succ_{\pi_1}(Pred_{\pi_1}(0) + 1)$. Finally, $y_{Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1)} = z_{Succ_{\pi_1}(Pred_{\pi_1}(0)+1)}$. From the definition of the contribution to the linking number we obtain :
$M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0) + 1) = M_{c,\pi_1}(i, Pred_{\pi_1}(0) + 1)$.

From the definition of $Succ_{\pi_1}(0)$ and $Pred_{\pi_1}(0)$ and the fact that the contribution of a couple (k, i) is equal to 0 in the case when $\mathcal{P}(z_k) = \mathcal{P}(z_{k-1})$ we have :

$$M_{c,\pi_1}(i, Pred_{\pi_1}(0) + 1) = \sum_{k=Pred_{\pi_1}(0)+1}^{p_1-1} M_{c,\pi_1}(i, k) + \sum_{k=0}^{Succ_{\pi_1}(0)-1} M_{c,\pi_1}(i, k).$$

Due to the expression of $L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i)$ set above, and due to the fact that the sequence of voxels of π_1 appears in $\pi_1.\pi_2$ between $Succ_{\pi_1.\pi_2}(0)$ and $Pred_{\pi_1.\pi_2}(0)$:

$$\begin{aligned}
L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) &= M_{c,\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0) + 1, k) + \sum_{k=Succ_{\pi_1.\pi_2}(0)}^{Pred_{\pi_1.\pi_2}(0)} M_{c,\pi_1.\pi_2}(i, k) \\
&= M_{c,\pi_1}(Pred_{\pi_1}(0) + 1, k) + \sum_{k=Succ_{\pi_1}(0)}^{Pred_{\pi_1}(0)} M_{c,\pi_1}(i, k) \\
&= L^{\pi_1}_{\pi_1,c}(i)
\end{aligned}$$

• The case when $Succ_{\pi_1}(0) = 0$ and $Succ_{\pi_2}(0) \neq 0$ is similar.

• In the case when none of the paths π_1 and π_2 has a projection reduced to a single point (i.e. $Succ_{\pi_1}(0) \neq 0$ and $Succ_{\pi_2}(0) \neq 0$).

Then, following the same considerations as in the previous case we show that :

$$\begin{aligned}
L^{\pi_1}_{\pi_1,c}(i) &= M_{c,\pi_1}(i, Pred_{\pi_1}(0) + 1) + \sum_{k=Succ_{\pi_1}(0)}^{Pred_{\pi_1}(0)} M_{c,\pi_1}(i, k) && (2.1) \\
L^{\pi_2}_{\pi_2,c}(i) &= M_{c,\pi_2}(i, Pred_{\pi_2}(0) + 1) + \sum_{k=Succ_{\pi_2}(0)}^{Pred_{\pi_2}(0)} M_{c,\pi_2}(i, k) && (2.2) \\
L^{\pi_1.\pi_2}_{\pi_1.\pi_2,c}(i) &= M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0) + 1) + \sum_{k=Succ_{\pi_1.\pi_2}(0)}^{Pred_{\pi_1.\pi_2}(p_1)} M_{c,\pi_1.\pi_2}(i, k) && (2.3) \\
&+ M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1) + 1) + \sum_{k=Succ_{\pi_1.\pi_2}(p_1)}^{Pred_{\pi_1.\pi_2}(0)} M_{c,\pi_1.\pi_2}(i, k)
\end{aligned}$$

For $k \in [Succ_{\pi_1}(0), Pred_{\pi_1}(0)] = [Succ_{\pi_1.\pi_2}(0), Pred_{\pi_1.\pi_2}(p_1)] \subset]0, p_1[$ we have $z_k = y_k$, $z_{k-1} = y_{k-1}$ and $z_{Succ_{\pi_1}(k)} = y_{Succ_{\pi_1.\pi_2}(k)}$, so $M_{c,\pi_1}(i,k) = M_{c,\pi_1.\pi_2}(i,k)$. So the sum in equation (2.1) is equal to the first sum in equation (2.3).

For $k \in [Succ_{\pi_2}(0), Pred_{\pi_2}(0)] = [Succ_{\pi_1.\pi_2}(p_1)-p_1, Pred_{\pi_1.\pi_2}(0)-p_1] \subset]0, p_2[$ we have $t_k = y_{k+p_1}$, $t_{k-1} = y_{k+p_1-1}$ and $t_{Succ_{\pi_1}(k)} = y_{Succ_{\pi_1.\pi_2}(k)+p_1}$, so $M_{c,\pi_2}(i,k) = W_{c,\pi_1.\pi_2}(i, k+p_1)$. Then, the sum in equation (2.2) is equal to the second sum in equation (2.3).

There remains to prove the equality $M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + M_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1) + M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$. But,

$M^-_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) = M^-_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$ since $z_{Pred_{\pi_1}(0)} = y_{Pred_{\pi_1.\pi_2}(p_1)}$.

$M^+_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) = M^+_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1)$ since $z_{Succ_{\pi_1}(Pred_{\pi_1}(0)+1)} = z_{Succ_{\pi_1}(0)} = y_{Succ_{\pi_1.\pi_2}(0)} = y_{Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(0)+1)}$.

$M^-_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M^-_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1)$ since $t_{Pred_{\pi_2}(0)} = y_{Pred_{\pi_1.\pi_2}(0)}$.

$M^+_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M^+_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$ since $t_{Succ_{\pi_2}(Pred_{\pi_2}(0)+1)} = t_{Succ_{\pi_2}(0)} = t_{Succ_{\pi_1.\pi_2}(p_1)} = t_{Succ_{\pi_1.\pi_2}(Pred_{\pi_1.\pi_2}(p_1)+1)}$.

Finally, $M_{c,\pi_1}(i, Pred_{\pi_1}(0)+1) + M_{c,\pi_2}(i, Pred_{\pi_2}(0)+1) = M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(0)+1) + M_{c,\pi_1.\pi_2}(i, Pred_{\pi_1.\pi_2}(p_1)+1)$. □

Proposition 11.3 *Let c_1 and c_2 be two closed paths with the same extremity and π be a closed path such that $\mathcal{H}(c_1,\pi)$ and $\mathcal{H}(c_1,\pi)$ hold. Then, $L_{\pi,c_1.c_2} = L_{\pi,c_1} + L_{\pi,c_2}$.*

Proof : Let $\pi = (y_0,\ldots,y_p)$, $c_1 = (z_0,\ldots,z_{q_1})$, $c_2 = (t_0,\ldots,t_{q_2})$ and $c_1.c_2 = (x_0,\ldots,x_{q_1+q_2})$. We prove that $\displaystyle\sum_{k=0}^{p-1} L^{c_1.c_2}_{\pi,c_1.c_2}(k) = \sum_{k=0}^{p-1} L^{c_1}_{\pi,c_1}(k) + \sum_{k=0}^{p-1} L^{c_2}_{\pi,c_2}(k)$.

More precisely we show that for any $k \in [0, p-1]$ we have :

$L^{c_1.c_2}_{\pi,c_1.c2}(k) = L^{c_1}_{\pi,c_1}(k) + L^{c_2}_{\pi,c_2}(k)$.

From the definition of the direct contribution of a couple (k,i) to the linking number, both terms of the previous equality are equal to zero if $\mathcal{P}(y_{k-1}) = \mathcal{P}(y_k)$ or if $\mathcal{P}(y_{k-1}) = \mathcal{P}(y_{Succ_\pi(k)})$. Therefore, we have to investigate the case when the projective movement $\mathcal{P} = P_\pi(k)$ (see Definition 11.4) is not trivial in the sense that $\mathcal{P}^{Pred} \neq \mathcal{P}^{Succ}$.

In this case, we prove that :

$$\sum_{i=0}^{q_1+q_2-1} W_{\pi,c_1.c_2}(k,i) = \sum_{i=0}^{q_1-1} W_{\pi,c_1}(k,i) + \sum_{i=0}^{q_2-1} W_{\pi,c_2}(k,i)$$

• In the case when both c_1 and c_2 are closed paths the projection of which is reduced to a single point, i.e. $\mathcal{P}(z_0) = \mathcal{P}(z_i)$ for any $i \in [0, q_1]$ and $\mathcal{P}(t_0) = \mathcal{P}(t_i)$ for any $i \in [0, q_2]$

then it is immediate that $L_{\pi,c_1} = L_{\pi,c_2} = L_{\pi,c_1.c_2} = 0$.

• In the case when the curve c_2 has a projection reduced to a single point (i.e. $Succ_{c_2}(0) = 0$ and c_1 has a projection which is not reduced to a single point (i.e. $Succ_{c_1}(0) \neq 0$). Then, $L_{\pi,c_1} = 0$ and we prove that $L_{\pi,c_2} = L_{\pi,c_1.c_2}$.

Indeed, in this case :

$$L_{\pi,c_1.c_2} = \sum_{i=0}^{Succ_{c_1.c_2}(0)-1} W_{\pi,c_1.c_2}(k,i) + \sum_{i=Succ_{c_1.c_2}(0)}^{Pred_{c_1.c_2}(0)} W_{\pi,c_1.c_2}(k,i) + \sum_{i=Pred_{c_1.c_2}(0)}^{q_1+q_2-1} W_{\pi,c_1.c_2}(k,i).$$

But from the definition of $Succ_{c_1.c_2}(0)$ and $Pred_{c_1.c_2}(0)$; from the definition of the contribution of a couple (k,i) in the case when $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ we obtain that

$$\sum_{i=Pred_{c_1.c_2}(0)}^{q_1+q_2-1} W_{\pi,c_1.c_2}(k,i) + \sum_{i=0}^{Succ_{c_1.c_2}(0)-1} W_{\pi,c_1.c_2}(k,i) = L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1).$$

We also observe that : $Pred_{c_1.c_2}(0) = Pred_{c_1}(0) \in]0,q_1[$ since $\mathcal{P}(x_j) = \mathcal{P}(x_{q_1})$ for $j = q_1,\ldots,q_1+q_2$. But $x_j = z_j$ for all $j \in [0, ldots, q_1]$ so :

– $x_{Pred_{c_1.c_2}(0)} = z_{Pred_{c_1}(0)}$ and $x_{Pred_{c_1.c_2}(0)+1} = z_{Pred_{c_1}(0)+1}$.

On the other hand, $Succ_{c_1.c_2}(Pred_{c_1.c_2}(0)+1) = Succ_{c_1.c_2}(0)$ from the definition of $Succ$ and $Pred$. But $Succ_{c_1.c_2}(0) \in]0,q_1[$, so $Succ_{c_1.c_2}(0) = Succ_{c_1}(0) = Succ_{c_1}(Pred_{c_1}(0)+1)$. Finally,

– $x_{Succ_{c_1.c_2}(Pred_{c_1.c_2}(0)+1)} = z_{Succ_{c_1}(Pred_{c_1}(0)+1)}$.

From the definition of the contribution to the linking number we obtain that $L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1) = L_{\pi,c_1}(k, Pred_{c_1}(0)+1)$. And from the definition of $Succ_{c_1}(0)$ and $Pred_{c_1}(0)$ and the contribution of a couple (k,i) in the case when $\mathcal{P}(z_i) = \mathcal{P}(t_{i-1})$ we have :

$$L_{\pi,c_1}(k, Pred_{c_1}(0)+1) = \sum_{i=Pred_{c_1}(0)}^{q_1-1} W_{\pi,c_1}(k,i) + \sum_{i=0}^{Succ_{c_1}(0)-1} W_{\pi,c_1}(k,i).$$

Let us compute the sum $\displaystyle\sum_{i=Succ_{c_1.c_2}(0)}^{Pred_{c_1.c_2}(0)} W_{\pi,c_1.c_2}(k,i)$.

For $i \in [Succ_{c_1.c_2}(0), Pred_{c_1.c_2}(0)] = [Succ_{c_1}(0), Succ_{c_1}(0)] \subset]0,q_1[$ we have : $x_i = z_i$, $x_{i-1} = z_{i-1}$ and $Succ_{c_1.c_2}(i) \leq Pred_{c_1.c_2}(0)+1 = Pred_{c_1}(0)+1 \leq q_1$ so that $x_{Succ_{c_1.c_2}(i)} = z_{Succ_{c_1}(i)}$. This implies that $W_{\pi,c_1.c_2}(k,i) = W_{\pi,c_1}(k,i)$ so

$$\sum_{i=Succ_{c_1.c_2}(0)}^{Pred_{c_1.c_2}(0)} W_{\pi,c_1.c_2}(k,i) = \sum_{i=Succ_{c_1}(0)}^{Pred_{c_1}(0)} W_{\pi,c_1}(k,i).$$

We obtain that :

$$L_{\pi,c_1.c_2} = \sum_{i=Pred_{c_1}(0)}^{q_1-1} W_{\pi,c_1}(k,i) + \sum_{i=0}^{Succ_{c_1}(0)-1} W_{\pi,c_1}(k,i) + \sum_{i=Succ_{c_1}(0)}^{Pred_{c_1}(0)} W_{\pi,c_1}(k,i) = L_{\pi,c_1}.$$

• The case when $Succ_{c_1}(0) = 0$ and $Succ_{c_2}(0) \neq 0$ can be proved by a very similar way with heavy notations in order to shift the subscripts of identical voxels of c_2 and $c_1.c_2$.

• In the case when no one of the paths c_1 and c_2 has a projection reduced to a single point (i.e. $Succ_{c_1}(0) \neq 0$ and $Succ_{c_2}(0) \neq 0$).

Then, following the same considerations as in the previous case we show that :

$$\begin{aligned}
L^{c_1}_{\pi,c_1}(k) &= L_{\pi,c_1}(k, Pred_{c_1}(0)+1) &&+ \sum_{i=Succ_{c_1}(0)}^{Pred_{c_1}(0)} W_{\pi,c_1}(k,i) \\
L^{c_2}_{\pi,c_2}(k) &= L_{\pi,c_2}(k, Pred_{c_2}(0)+1) &&+ \sum_{i=Succ_{c_2}(0)}^{Pred_{c_2}(0)} W_{\pi,c_2}(k,i) \\
L^{c_1.c_2}_{\pi,c_1.c_2}(k) &= L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1) &&+ \sum_{i=Succ_{c_1.c_2}(0)}^{Pred_{c_1.c_2}(q_1)} W_{\pi,c_1.c_2}(k,i) \\
&+ L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(q_1)+1) &&+ \sum_{i=Succ_{c_1.c_2}(q_1)}^{Pred_{c_1.c_2}(0)} W_{\pi,c_1.c_2}(k,i)
\end{aligned}$$

For $i \in [Succ_{c_1}(0), Pred_{c_1}(0)] = [Succ_{c_1.c_2}(0), Pred_{c_1.c_2}(q_1)] \subset]0, q_1[$ we have $z_i = x_i$, $z_{i-1} = x_{i-1}$ and $z_{Succ_{c_1}(i)} = x_{Succ_{c_1.c_2}(i)}$, so $L_{\pi,c_1}(k,i) = L_{\pi,c_1.c_2}(k,i)$. Then,

$$\sum_{i=Succ_{c_1}(0)}^{Pred_{c_1}(0)} W_{\pi,c_1}(k,i) = \sum_{i=Succ_{c_1.c_2}(0)}^{Pred_{c_1.c_2}(q_1)} W_{\pi,c_1.c_2}(k,i).$$

For $i \in [Succ_{c_2}(0), Pred_{c_2}(0)] = [Succ_{c_1.c_2}(q_1) - q_1, Pred_{c_1.c_2}(0) - q_1] \subset]0, q_2[$ we have $t_i = x_{i+q_1}$, $t_{i-1} = x_{i+q_1-1}$ and $t_{Succ_{c_1}(i)} = x_{Succ_{c_1.c_2}(i)+q_1}$, so $L_{\pi,c_2}(k,i) = L_{\pi,c_1.c_2}(k, i+q_1)$.

Then, $\displaystyle\sum_{i=Succ_{c_2}(0)}^{Pred_{c_2}(0)} W_{\pi,c_2}(k,i) = \sum_{i=Succ_{c_1.c_2}(q_1)}^{Pred_{c_1.c_2}(0)} W_{\pi,c_1.c_2}(k,i).$

On the other hand, $L_{\pi,c_1}(k, Pred_{c_1}(0)+1) + L_{\pi,c_2}(k, Pred_{c_2}(0)+1) = L^-_{\pi,c_1}(k, Pred_{c_1}(0)+1) + L^+_{\pi,c_1}(k, Pred_{c_1}(0)+1) + L^-_{\pi,c_2}(k, Pred_{c_2}(0)+1) + L^+_{\pi,c_2}(k, Pred_{c_2}(0)+1)$.

But,

$L^-_{\pi,c_1}(k, Pred_{c_1}(0)+1) = L^-_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(q_1)+1)$ since $z_{Pred_{c_1}(0)} = x_{Pred_{c_1.c_2}(q_1)}$.

$L^+_{\pi,c_1}(k, Pred_{c_1}(0)+1) = L^+_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1)$ since $z_{Succ_{c_1}(Pred_{c_1}(0)+1)} = z_{Succ_{c_1}(0)} = x_{Succ_{c_1.c_2}(0)} = x_{Succ_{c_1.c_2}(Pred_{c_1.c_2}(0)+1)}$.

$L^-_{\pi,c_2}(k, Pred_{c_2}(0)+1) = L^-_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1)$ since $t_{Pred_{c_2}(0)} = x_{Pred_{c_1.c_2}(0)}$.

$L^+_{\pi,c_2}(k, Pred_{c_2}(0)+1) = L^+_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(q_1)+1)$ since $t_{Succ_{c_2}(Pred_{c_2}(0)+1)} = t_{Succ_{c_2}(0)} = t_{Succ_{c_1.c_2}(q_1)} = t_{Succ_{c_1.c_2}(Pred_{c_1.c_2}(q_1)+1)}$.

Finally, $L_{\pi,c_1}(k, Pred_{c_1}(0)+1) + L_{\pi,c_2}(k, Pred_{c_2}(0)+1) = L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1) + L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(q_1)+1)$.

Then,

$$\begin{aligned}
L^{c_1}_{\pi,c_1}(k) + L^{c_2}_{\pi,c_2}(k) &= L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(0)+1) + L_{\pi,c_1.c_2}(k, Pred_{c_1.c_2}(q_1)+1) \\
&\quad + \sum_{i=Succ_{c_1.c_2}(0)}^{Pred_{c_1.c_2}(q_1)} W_{\pi,c_1.c_2}(k,i) + \sum_{i=Succ_{c_1.c_2}(q_1)}^{Pred_{c_1.c_2}(0)} W_{\pi,c_1.c_2}(k,i) \\
&= L^{c_1.c_2}_{\pi,c_1.c_2}(k)
\end{aligned}$$

□

11.5 Proofs of the main theorems

11.5.1 Independence up to a deformation of the $6/(6+)-$path

In this section we will prove Theorem 14 in the case when $(n, \overline{n}) \in \{(6+, 18), (6, 26)\}$. The main idea of the proof is that a homotopic deformation of $6-$paths or $(6+)-$paths can be achieved by insertion/deletion of simple closed loops included in a cube or a square (like depicted in Figure 11.10). Then, proving that such small $n-$paths have a linking number of 0 with any other $\overline{n}-$path will be sufficient to prove the main theorem by using Proposition 11.2.

Definition 11.7 *Let π and π' be two closed $n-$paths ($n \in \{6, 6+, 18, 26\}$) in $X \subset \mathbb{Z}^3$. We say that π and π' are the* same up to a simple $n-$loop insertion/deletion *if $\pi = \pi_1.(p).\pi_2$ where p is a voxel, and $\pi' = \pi_1.\gamma.\pi_2$ where γ is a simple closed $n-$path from p to p included in a 2×2×2 cube (a 2×2 square if $(n, \overline{n}) = (6, 26)$); or if $\pi = \pi_1.\gamma.\pi_2$ and $\pi' = \pi_1.(p).\pi_2$.*

Proposition 11.4 *Let π and π' be two $n-$paths ($n \in \{6, 6+, 18, 26\}$) of $X \subset \mathbb{Z}^3$. Then the two following properties are equivalent :*

i) π is $n-$homotopic to π'.

ii) There exists a sequence $S = (\pi^0, \ldots, \pi^l)$ such that $\pi^0 = \pi$, $\pi^l = \pi'$ and for $h = 1 \ldots l$, the two paths π^{h-1} and π^h are the same up to a simple $n-$loop insertion/deletion

In case ii) is satisfied, we denote $\pi \equiv_{SL} \pi'$.

Proof :

$ii) \Rightarrow i)$ is obvious from the definitions since a simple $n-$loop insertion/deletion is a kind of elementary $n-$deformation. Conversely, it is sufficient to prove $ii)$ assuming that π and π' are the same up to an elementary $n-$deformation. So, suppose that $\pi = \pi_1.\gamma.\pi_2$ and $\pi' = \pi_1.\gamma'.\pi_2$. Where γ and γ' have the same extremities (say p and q) and are included in a 2×2×2 cube $\mathcal{C}$ (a 2×2 square if $(n, \overline{n}) = (6, 26)$).

We give the sequence S : First, by inserting or deleting simple loops of the form (x, y, x) in π we get : $\pi = \pi_1.\gamma.\pi_2 \equiv_{SL} \pi_1.\gamma.\gamma'^{-1}.\gamma'.\pi_2$. Now, consider the closed path $\gamma.\gamma'^{-1}$ from p to p. One can sequentially remove minimal sub-paths of the loop $\gamma.\gamma'^{-1}$ which are simple loops until the resulting path is itself a simple loop and can be fully removed. Finally, $\pi \equiv_{SL} \pi_1.\gamma.\gamma'^{-1}.\gamma'.\pi_2 \equiv_{SL} \pi_1.\gamma'.\pi_2 = \pi'$. □

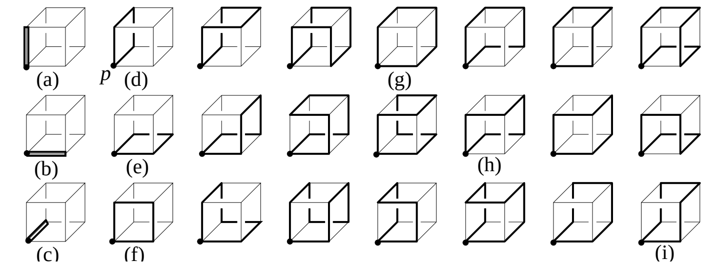


Figure 11.10: The 24 closed 6−loops from p in a 2×2×2 cube.

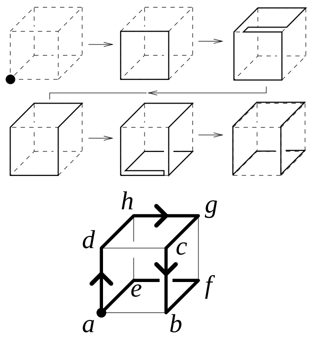


Figure 11.11: How to obtain a kind of simple 6−loop by insertion/deletion of simpler ones.

Lemma 11.5 *Let π be a simple $6-$loop included in a 2×2 square and c be a $26-$path such that $\mathcal{H}(\pi, c)$ holds, then $L_{\pi,c} = 0$.*

Proof : From Figure 11.10(a) to Figure 11.10(f), all the possible simple 6−loops included in a 2×2 square from a given point p to p are depicted up to a choice of an orientation. In this proof, we will only investigate the cases of simple loops from a fixed given point p. By changing the parameterization of the loop, it is clear that the proof is similar for the 3 other positions of the voxel p in a 2×2 square.

– Cases of the Figures 11.10(a) and 11.10(b) : In this case, $\pi = (x, y, x)$ and $L_{\pi,c} = L^c_{\pi,c}(0) + L^c_{\pi,c}(1) = 0$ from the very definition of the contribution to the linking number.

– Case of Figure 11.10(c) : In this case, $\pi = (x, y, x)$ and $\mathcal{P}(x) = \mathcal{P}(y)$ so it is clear that $L_{\pi,c} = 0$.

– Cases of the Figures 11.10(d) and 11.10(e) :

Let $\pi = (y_0, y_1, y_2, y_3, y_4 = y_0)$. In both cases and for any choice of a parameterization, one can easily check that $L^c_{\pi,c}(k) = 0$ for $k = 0, \ldots, 3$ either because $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ or because $P_\pi(k)^{Pred} = P_\pi(k)^{Succ}$.

– Case of Figure 11.10(f) :

We set $c = (x_i)_{i=0,\ldots,q}$, $\pi = (y_0, y_1, y_2, y_3, y_4 = y_0)$, $P = \{\mathcal{P}(y_0), \mathcal{P}(y_1), \mathcal{P}(y_2), \mathcal{P}(y_3)\}$ and let $I = \{[i_1, i_2] |\ \mathcal{P}(x_{i_1 - 1}) \notin P,\ \mathcal{P}(x_{i_2+1}) \notin P \text{ and } \mathcal{P}(x_i) \in P \text{ for all } i \in [i_1, i_2]\}$. In the case when $\mathcal{P}(x_i) \in P$ for all $i \in [0, q]$ then $I = \{[0, q-1]\}$.

Then, it is clear that $L_{\pi,c} = \displaystyle\sum_{[i_1,i_2]\in I} \sum_{i=i_1}^{i_2} L^\pi_{\pi,c}(i)$.

It is sufficient to prove that for any $[i_1, i_2] \in I$ the sum $\displaystyle\sum_{i=i_1}^{i_2} L^\pi_{\pi,c}(i)$ is equal to 0. One can choose an orientation for the path π but in all cases we observe that for all $i \in [i_1, i_2]$ there exists a single subscript $k(i) \in \{0, 1, 2, 3\}$ such that $L^\pi_{\pi,c}(i) = W_{\pi,c}(k(i), i)$. First, for a given interval $[i_1, i_2]$, either $x^3_i < y^3_{k(i)}$ for all $i \in [i_1, i_2]$ or $x^3_i > y^3_{k(i)}$ for all $i \in [i_1, i_2]$. In the first case, $L^\pi_{\pi,c}(i) = 0$ for all $i \in [i_1, i_2]$ and there is nothing else to prove. In the second case one can consider some $a_0 < a_1 < \ldots < a_l \leq a_{l+1}$ such that : $[i_1, i_2] = [a_0, a_1[\cup [a_1, a_2[\cup \ldots \cup [a_l, a_{l+1}]$, with $\mathcal{P}(x_{a_i}) \neq \mathcal{P}(x_{a_i - 1})$ for any $i \in \{1, \ldots, l\}$; and $\forall i \in \{0, \ldots, l\}$, $\forall j \in [a_i, a_{i+1}[$ we have $\mathcal{P}(x_j) = \mathcal{P}(x_{a_i})$. Then, for any $i \in \{0, \ldots, l-1\}$ we have :

$$\sum_{j=a_i}^{a_{i+1}-1} W_{\pi,c}(k(j), j) = W_{\pi,c}(k(a_i), a_i) \text{ and } \sum_{j=a_l}^{a_{l+1}} W_{\pi,c}(k(j), j) = W_{\pi,c}(k(a_l), a_l).$$

By construction of the intervals $[a_i, a_{i+1}[$, we have :

$$\sum_{i\in[i_1,i_2]} W_{\pi,c}(k(i), i) = \sum_{i=1,\ldots,l} W_{\pi,c}(k(a_i), a_i).$$

Now, we prove that for $i = 1, \ldots, l$, $W^+_{\pi,c}(k(a_i), a_i) + W^-_{\pi,c}(k(a_{i+1}), a_{i+1}) = 0$. Indeed, let $\beta_i = P_c(a_i)$ and $\alpha_i = P_\pi(k(a_i))$. Then $W^+_{\pi,c}(k(a_i), a_i)$ depends on the position of the point β^{Succ}_i with respect to the projective movement α_i. If $\beta^{Succ}_i \in \{\alpha^{Pred}, \alpha^{Succ}\}$ then we have $W^+_{\pi,c}(k(a_i), a_i) = 0$ and $W^-_{\pi,c}(k(a_{i+1}), a_{i+1}) = 0$ since $Pred_c(a_{i+1}) = a_i$. If $\beta^{Succ}_i \notin \{\alpha^{Pred}, \alpha^{Succ}\}$ then β^{Succ} is a point which is 8−adjacent but not 4−adjacent to $(0, 0)$. But in this case, and depending on the orientation of the loop π, if $\beta^{Succ}_i \in Right(\alpha_i)$ then $\beta^{Pred}_{i+1} \in Right(\alpha_{i+1})$ and if $\beta^{Succ}_i \in Left(\alpha_i)$ then $\beta^{Pred}_{i+1} \in Left(\alpha_{i+1})$.

We also see that $W^-_{\pi,c}(k(a_0), a_0) + W^+_{\pi,c}(k(a_l), a_l) = 0$. Indeed, up to a choice of an orientation for the loop reduced to a 2×2 square, it is clear that the projections of the voxels $P_c(a_0)^{Pred}$ and $P_c(a_l)^{Succ}$ will both belong either respectively to $Right(P_\pi(k(a_0)))$ and $Right(P_\pi(k(a_l)))$ or to $Left(P_\pi(k(a_0)))$ and $Left(P_\pi(k(a_l)))$. In the case when $I = \{[0, q-1]\}$ then $W^-_{\pi,c}(k(a_0), a_0) + W^+_{\pi,c}(k(a_l), a_l) = 0$ either because both terms are equal to 0 or for the same reason as explained above for the intervals $[a_i, a_{i+1}[$ (contributions with opposite signs).

Finally, for any $[i_1, i_2] \in I$,

$$\begin{aligned}
\sum_{i=i_1}^{i_2} L^\pi_{\pi,c}(i) &= \sum_{i=1}^{l} \left(W^-_{\pi,c}(k(a_i), a_i) + W^+_{\pi,c}(k(a_i), a_i)\right) \\
&= W^-_{\pi,c}(k(a_1), a_1) \ + \ W^+_{\pi,c}(k(a_l), a_l) \ + \\
&\quad \sum_{i=1}^{l} \left(W^+_{\pi,c}(k(a_i), a_i) W^-_{\pi,c}(k(a_{i+1}), a_{i+1})\right) \\
&= 0
\end{aligned}$$

□

Lemma 11.6 *If π is a simple $(6+)-$loop included in a 2×2×2 cube and c is a closed $18-$path such that $\mathcal{H}(\pi, c)$ holds, then $L_{\pi,c} = 0$.*

Proof : In Figure 11.10 are depicted up to a choice of a parameterization and for a given point p all the simple loops in a 2×2×2 cube from p to p (the proof when p is any one of the 7 other voxels in the cube is similar).

In this proof we only have to show that $L_{\pi,c} = 0$ when π is one of the loops $(a) \ldots (i)$. Then, using Proposition 11.2 and the fact that any other loop of Figure 11.10 can be obtained by insertion/deletion of loops $(a) \ldots (i)$ we will achieve to prove that $L_{\pi,c} = 0$ when π is any of the loops of Figure 11.10.

In the proof of Lemma 11.5, we have already proved that $L_{\pi,c} = 0$ when π is one of the loops of figures (a), (b), (c), (d), (e) and (f) (indeed, c, as an $18-$path, is also a $26-$path).

– Cases of the Figures 11.10(g), (h) and (i) :

The proof in these cases is similar to the case of Figure 11.10(f) (see Proof of Proposition 11.5). Thus, we still observe the existence of an integer $k(i)$ such that $L^\pi_{\pi,c}(i) = W_{\pi,c}(k(i), i)$. Indeed, only one of any two voxels of π which have the same projection may have a non zero contribution. We also still use the fact that for a given interval $[i_1, i_2] \in I$

(as defined in the previous case), either $x_i^3 < y_{k(i)}^3$ for all $i \in [i_1, i_2]$ or $x_i^3 > y_{k(i)}^3$ for all $i \in [i_1, i_2]$. Indeed, this comes from the fact that the two points of the cube which are not points of π are not 18−adjacent, so that the path c cannot have intersection intervals with voxels in the two sides of π according to their third coordinates. The end of the proof is similar.

– Other cases : Now, if π is a simple closed loop in $\mathcal{C}$ such that $L_{\pi,c} = 0$ and π' is a simple loop in $\mathcal{C}$ obtained by the insertion of any of the loops $(a), \ldots, (i)$ in π, then $L_{\pi,c} = L_{\pi',c} = 0$. Indeed, if $\pi = \pi_1.(x).\pi_2$ and $\pi' = \pi_1.\gamma.\pi_2$ where γ is a loop from x to x of some form in $(a), \ldots, (i)$, then $L_{\pi_1.\gamma.\pi_2,c} = L_{\pi_2.\pi_1.\gamma,c}$ because the linking number is invariant under any orientation preserving change of parameterization. Furthermore, from Proposition 11.2 we have $L_{\pi_2.\pi_1.\gamma,c} = L_{\pi_2.\pi_1,c} + L_{\gamma,c}$. Since we have proved that $L_{\gamma,c} = 0$ when γ is of type $(a), \ldots, (i)$ then $L_{\pi_2.\pi_1.\gamma,c} = L_{\pi_2.\pi_1,c}$ and again we have $L_{\pi_2.\pi_1,c} = L_{\pi_1.\pi_2,c}$. Finally, $L_{\pi',c} = L_{\pi,c}$.

Now, it is left to the reader to check that any of the 15 other simple loops can be obtained by a sequence of insertions or deletions of loops like $(a), \ldots, (i)$. Then, we obtain that $L_{\pi,c} = 0$ when π is any kind of loop depicted in Figure 11.10.

As an example, we only give here the sequence of simple loops insertion/deletion of the kind $(a) \ldots (i)$ which leads from the path reduced to the voxel a to the path depicted in Figure 11.11.

$(\underline{a, d, c, b, a})$ can be obtained from $(\underline{a})$ by insertion of a loop like Figure 11.10(f).

$(a, \underline{d, h, g, c, d}, c, b, a)$ can be obtained from $(a, \underline{d}, c, b, a)$ by insertion of a loop like Figure 11.10(e).

$(a, d, h, g, \underline{c}, b, a)$ can be obtained from $(a, d, h, g, \underline{c, d, c}, b, a)$ by deletion of a loop like Figure 11.10(b).

$(a, d, h, g, c, \underline{b, f, e, a, b}, a)$ can be obtained from $(a, d, h, g, c, \underline{b}, a)$ by insertion of a loop like Figure 11.10(e).

$(a, d, h, g, c, b, f, e, \underline{a})$ can be obtained from $(a, d, h, g, c, b, f, e, \underline{a, b, a})$ by deletion of a loop like Figure 11.10(b). $\square$

Proof of Theorem 14 : From Proposition 11.4 it is sufficient to prove Theorem 14 in the case when π and π' are the same up to a simple n−loop insertion/deletion ($n \in \{6, 6+\}$). In this case, let us suppose that $\pi = \pi_1.(x).\pi_2$ and $\pi' = \pi_1.\gamma.\pi_2$ where γ is a simple loop from x to x included in a 2×2×2 cube $\mathcal{C}$ (in a 2×2 square if $(n, \overline{n}) = (6, 26)$). Since the linking number is invariant under any order preserving change of parameterization, we have $L_{\pi',c} = L_{\pi_1.\gamma.\pi_2,c} = L_{\gamma.\pi_2.\pi_1,c}$. From Proposition 11.2, $L_{\gamma.\pi_2.\pi_1,c} = L_{\gamma,c} + L_{\pi_2.\pi_1,c}$ ($\pi_2.\pi_1$ is a closed $n-$path from x to x as well as γ). Now, since γ is a simple loop in $\mathcal{C}$ and from Lemma 11.5 or Lemma 11.6 we have $L_{\gamma,c} = 0$. Finally, $L_{\gamma.\pi_2.\pi_1,c} = L_{\pi_2.\pi_1,c} = L_{\pi_1.\pi_2,c} = L_{\pi,c}$. □

11.5.2 Independence up to a deformation of the $26-$path

Definition 11.8 *Let c and c' be two $26-$paths in $X \subset \mathbb{Z}^3$. We say that c and c' are* the same up to a triangle or a back and forth insertion *if :*

– Either $c = c_1.(x).c_2$ and $c' = c_1.(x,y,z,x).c_2$ where the voxels x, y and z are included in a 2×2×2 cube $\mathcal{C}$,

– or $c = c_1.(x).c_2$ and $c' = c_1.(x,y,x).c_2$.

We say that c and c' are the same up to a triangle or a back and forth insertion/deletion if either c and c' or c' and c are the same up to a triangle or a back and forth insertion.

Proposition 11.7 *Let c and c' be two $26-$paths in $X \subset \mathbb{Z}^3$. Then the two following properties are equivalent :*

i) c is $26-$homotopic to c' in X.

ii) There exists a sequence $S = (c^0 = c, \dots, c^k = c')$ of paths in X such that for all $i \in [1, k[$ the paths c^{i-1} and c^i are the same up to a triangle or back and forth insertion/deletion.

If ii) is satisfied, we denote $c \equiv_{TBF} c'$.

Proof : *ii)* $\Rightarrow$ *i')* is obvious since a triangle or back and forth insertion is a particular case of elementary $26-$deformation.

i') $\Rightarrow$ *ii)* Conversely, from Definition 2.11, it is sufficient to prove *ii)* if c and c' are the same up to an elementary deformation. We suppose that $c = c_1.\gamma.c_2$ and $c' = c_1.\gamma'.c_2$ where γ and γ' have the same extremities and are included in a 2×2×2 cube $\mathcal{C}$. By an induction on the length of γ we show that there exists a sequence of triangles or back and forth insertions/deletions which leads from γ to the path reduced to its extremities. Suppose that $\gamma = \gamma^0$ has a length $l \geq 2$. Then we have $\gamma^0 = \gamma_1^0.(x,y,z).\gamma_2^0$. Now, by a back and forth insertion we can obtain the path $\gamma_1^0.(x,y,z).(z,x,z).\gamma_2^0 = \gamma_1^0.(x,y,z,x,z).\gamma_2^0$ and then by a triangle deletion we obtain the path $\gamma_1^0.(x,z).\gamma_2^0 = \gamma^1$ which has a length of $l-1 < l$. By induction, we can obtain a path $\gamma^k = (p,q)$ with a length of 1 where p and q are the common extremities of γ and γ'.

By the same way we can obtain the path γ' from the path (p,q) by a sequence of triangle insertions and back and forth deletions. Finally, any elementary $26-$deformation can be done by a sequence of triangle or back and forth insertions/deletions. □

Lemma 11.8 *If c is a $26-$triangle, then for any $6-$path π such that $\mathcal{H}(\pi,c)$ holds we have $L_{\pi,c} = 0$.*

Proof : Let c be a $26-$triangle. We first consider the case when exactly two voxels of c have the same projection (the case when all the voxels have the same projection immediately implies that $L_{\pi,c} = 0$).

We suppose that two voxels of $c = (x_0, x_1, x_2, x_0)$ have the same projection. Without loss of generality, we suppose that $\mathcal{P}(x_1) = \mathcal{P}(x_2)$. Now, for any $6-$path π we have $L_{\pi,c} = L^{\pi}_{\pi,c}(0) + L^{\pi}_{\pi,c}(1) + L^{\pi}_{\pi,c}(2)$. But from Definition 11.4 we have $L^{\pi}_{\pi,c}(0) = 0$ since $Succ_c(0) = 1$ and $Pred_c(0) = 2$ and $\mathcal{P}(x_1) = \mathcal{P}(x_2)$. We also have $L^{\pi}_{\pi,c}(1) = 0$ since $Succ_c(1) = Pred_c(1) = 0$. Finally, $L^{\pi}_{\pi,c}(2) = 0$ since $\mathcal{P}(x_1) = \mathcal{P}(x_2)$.

Now, we assume that the three voxels of c have pairwise distinct projections.

Let $c = (x_0, x_1, x_2, x_3 = x_0)$ and $\pi = (y_k)_{k=0,\dots,p}$. Let $P = \{\mathcal{P}(x_0), \mathcal{P}(x_1), \mathcal{P}(x_2)\}$ and $K = \{[k_1,k_2] |\ \mathcal{P}(y_{k_1-1}) \notin P,\ \mathcal{P}(y_{k_2+1}) \notin P$ and $\mathcal{P}(y_i) \in P$ for all $i \in [k_1,k_2]\}$. If $\mathcal{P}(y_k) \in P$ for all $k \in [0,p]$ then $K = \{[0,p-1]\}$. It is clear that $L_{\pi,c} = \displaystyle\sum_{[k_1,k_2]\in K} \sum_{k=k_1}^{k_2} L^{c}_{\pi,c}(k)$.

For any $[k_1,k_2] \in K$ and any $k \in [k_1,k_2]$ we denote by $i(k)$ the only subscript of voxels of c such that $\mathcal{P}(x_{i(k)}) = \mathcal{P}(y_k)$. Then, for any such k, we have $L^{c}_{\pi,c}(k) = W_{\pi,c}(k, i(k))$. So, $L_{\pi,c} = \displaystyle\sum_{[k_1,k_2]\in K} \sum_{k=k_1}^{k_2} W_{\pi,c}(k, i(k))$.

Now, from the definition of the contribution to the linking number, it is clear that for any $[k_1,k_2] \in K$: $\displaystyle\sum_{k=k_1}^{k_2} W_{\pi,c}(k,i(k)) = \sum_{k=k_1}^{Pred_\pi(k_2)+1} W_{\pi,c}(k,i(k))$.

But for $k \in [k_1+1, Pred_\pi(k_2)]$, the contribution $W_{\pi,c}(k,i(k))$ is equal to 0 either because $\{P_\pi(k)^{Pred}, P_\pi(k)^{Succ}\} \subset \{P_c(i(k))^{Pred}, P_c(i(k))^{Succ}\}$ or because $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$.

Similarly, we observe that $W^{+}_{\pi,c}(k_1, i(k_1)) = W^{-}_{\pi,c}(Pred_\pi(k_2)+1, i(Pred_\pi(k_2)+1)) = 0$. On the other hand, $W^{-}_{\pi,c}(k_1, i(k_1)) + W^{+}_{\pi,c}(Pred_\pi(k_2)+1, i(Pred_\pi(k_2)+1)) = 0$. Indeed, depending on a choice of an orientation for the triangle c, it is clear that the projections of the voxels $P_\pi(a_1)^{Pred}$ and $P_\pi(Pred_\pi(k_2)+1)^{Succ}$ will either belong respectively to $Right(P_c(i(a_1)))$ and $Right(P_c(i(Pred_\pi(k_2)+1)))$ or belong respectively to $Left(P_c(i(a_1)))$ and $Left(P_c(i(Pred_\pi(k_2)+1)))$. If $K=\{[0,p-1]\}$ then $W^-_{\pi,c}(0,i(0)) = W^+_{\pi,c}(Pred_\pi(p)+1, i(Pred_\pi(p)+1)) = 0$ from the definition of $W_{\pi,c}(k,i)$. Thus,

$$\begin{aligned}\sum_{k=k_1}^{k_2} W_{\pi,c}(k,i(k)) &= W_{\pi,c}(k_1,i(k_1)) + \sum_{k=k_1+1}^{Pred_\pi(k_2)} W_{\pi,c}(k,ik)) \\ &\quad + W_{\pi,c}(Pred_\pi(k_2)+1, i(Pred_\pi(k_2)+1)) = 0\end{aligned}$$

and finally $L_{\pi,c}=0$. □

Proof of Theorem 15 in the case $(6,26)$ **:** The proof is similar to the proof of Theorem 14 using Proposition 11.7 instead of Proposition 11.4 and Proposition 11.3 instead of Proposition 11.2. Lemma 11.8 shows that $L_{\pi,\gamma}=0$ when γ is a $26-$triangle. The case when γ is a back and forth is obvious and we also have $L_{\pi,\gamma}=0$ in this case. □

11.5.3 Independence up to a deformation of the $18-$path

Definition 11.9 *Let c and c' be two closed $18-$paths in $X\subset\mathbb{Z}^3$. We say that c and c' are* the same up to a triangle, back and forth or square insertion *respectively if :*

– *$c=c_1.(x).c_2$ and $c'=c_1.(x,y,z,x).c_2$ where the voxels x, y and z are included in a $2\times2\times2$ cube $\mathcal{C}$,*

– *$c=c_1.(x).c_2$ and $c'=c_1.(x,y,x).c_2$,*

– *$c=c_1.(x).c_2$ and $c'=c_1.\gamma.c_2$ where γ is one of the closed paths depicted in Figure 11.13 (up to a parameterization).*

We say that c and c' are the same up to a triangle, back and forth or square insertion/deletion if either c and c' or c' and c are the same up to a triangle, back and forth or square insertion.

Proposition 11.9 *Let c and c' be two closed $18-$paths in $X\subset\mathbb{Z}^3$. Then the two following properties are equivalent :*

i) c is $18-$homotopic to c' in X.

ii) There exists a sequence $S=(c^0,\dots,c^k)$ of paths in X with $c^0=c$ and $c^k=c'$, such that for all $i\in[1,k]$, the paths c^{i-1} and c^i are the same up to a triangle, back and forth or square insertion/deletion.

In case ii) is satisfied we denote $c \equiv_{TBS} c'$.

Proof : *i)* $\Leftarrow$ *ii)* is obvious since the insertion/deletion of a back and forth, a triangle or a square is a particular case of elementary $18-$deformation.

i) $\Rightarrow$ *ii)* Conversely, from Proposition 11.4, if c' and c are $18-$homotopic, then there exists a sequence of simple $18-$loops insertion/deletion which leads from c to c'. We prove that each step of simple $18-$loop insertion/deletion can be achieved by a sequence of back and forth, triangle or square insertions/deletions.

Let γ be a simple $18-$loop in a 2×2×2 cube. By an induction on the length of γ, we show that γ can be reduced to a single voxel by a sequence of back and forth, triangle or square insertions/deletions. This will indeed prove that any simple $18-$loop can be obtained by a sequence of insertion/deletion of back and forth, squares or triangles in a path reduced to a single voxel. Note that each step of this sequence only involves voxels of the loop γ so that the intermediate paths do belong to X.

Let $\gamma = \gamma^0$ be any simple closed $18-$loop, then given γ^k we must distinguish several cases :

• γ^k is reduced to a single voxel, there is nothing to prove in this case.

• If γ^k has a length 2, say $\gamma^k = (x, y, x)$, then $\gamma^{k+1} = (x)$ can be obtained by a back and forth deletion.

• If γ^k has a length 3, say $\gamma^k = (x, y, z, x)$, then $\gamma^{k+1} = (x)$ can be obtained by a triangle deletion.

• If γ^k has a length $l > 3$, then we distinguish two cases :

– If there exists x, y and z in γ^k such that $\gamma^k = \gamma_1^k.(x, y, z).\gamma_2^k$ where x is $18-$adjacent to z. In this case, the path $\gamma_1^k.(x, y, z).(z, x, z).\gamma_2^k = \gamma_1^k.(x, y, z, x, z).\gamma_2^k$ can be obtained by a back and forth insertion and then the path $\gamma^{k+1} = \gamma_1^k.(x, z).\gamma_2^k$ is obtained by a triangle deletion. The path γ^{k+1} has a length equal to $l - 1$.

– If there exists no subsequence (x, y, z) in γ^k such that x is $18-$adjacent to z. Then, in this case we prove that γ is a $18-$square (i.e. one of the loops of Figure 11.13). Indeed, we have depicted in Figure 11.12 a 2×2×2 cube. Suppose that γ has a length $l > 3$ and no triangle. Let us consider any two consecutive voxels of γ; up to a rotation these two voxels may have the configuration of the couple (a, b) or (a, h) of Figure 11.12. First, suppose that the two consecutive voxels have the same configuration as a and b in Figure 11.12 and try to extend this part of a simple $18-$loop taking care not to add a voxel which would be 18−adjacent to the predecessor of its predecessor. Then, the only kind of loop you can obtain is the loop depicted in Figure 11.13(b). By the same way, trying to extend the sequence (a, h) into a simple 18−loop will also lead to the path depicted in Figure 11.13(b). Finally, γ^k is a square which obviously can be removed by a square deletion in order to obtain a path γ^{k+1} reduced to a single voxel.

In all cases, we can obtain a path γ^{k+1} with a length either lower than l or equal to 1 by insertion/deletion of back and forth, triangles or squares. By induction, it is clear that there must exist an integer h such that γ^h is reduced to a single voxel. Then, any simple 18−loop can be inserted of removed in a closed 18−path by a sequence of triangles, back and forth or squares insertions/deletions. This achieve to prove that $i) \Rightarrow ii)$.

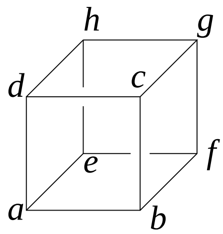


Figure 11.12: A 2×2×2 cube.

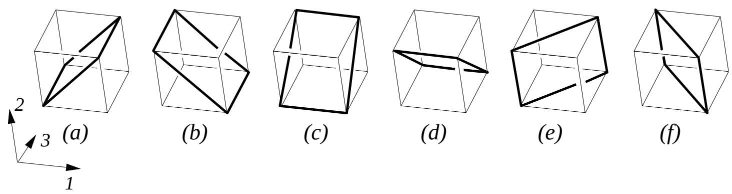


Figure 11.13: Possible simple 18−loops with no triangle in a 2×2×2 cube.

□

Lemma 11.10 *If c is a 18−square and π is a closed (6+)−path such that $\mathcal{H}(\pi, c)$ holds, then $L_{\pi,c} = 0$.*

Proof : Let $c = (x_0, x_1, x_2, x_3, x_0)$.

– Case of a square of the kind depicted in Figure 11.13(a) and Figure 11.13(b).

In this case, we have $L_{\pi,c} = L^{\pi}_{\pi,c}(0) + L^{\pi}_{\pi,c}(1) + L^{\pi}_{\pi,c}(2) + L^{\pi}_{\pi,c}(3)$ but for $i \in \{0, 1, 2, 3\}$ $L^{\pi}_{\pi,c}(i) = 0$ either since $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ or since $P_c(i)^{Pred} = P_c(i)^{Succ}$.

– Case of a square of the kind depicted in Figures 11.13(c), 11.13(d), 11.13(e) and 11.13(f).

The proof of the lemma in these cases is similar to the case of Figure 11.10(f) in the proof of Lemma 11.5 but a little less tricky since the case when two consecutive voxels of π have 8−adjacent projections which are not 4−adjacent cannot occur since c is a 6−path or a (6+)−path. Note that, with respect to Lemma 11.5, we must use $M_{c,\pi}$ here, instead of $W_{\pi,c}$ as in Lemma 11.5. □

Proof of Theorem 15 in the case $(6,18)$ **:** Again, the proof is similar to the proof of Theorem 14 using Proposition 11.9 instead of Proposition 11.4 and Proposition 11.3 instead of Proposition 11.2. Lemma 11.8 shows that $L_{\pi,\gamma} = 0$ when γ is a $18-$triangle (which is also a $26-$triangle). The case when γ is a back and forth is obvious and we also have $L_{\pi,\gamma} = 0$. Finally, Lemma 11.10 is used to prove that $L_{\pi,\gamma} = 0$ when γ is a square. $\square$

Chapter 12

A concise characterization of 3D simple points

This chapter will state the main result of this part which is that a not less restrictive criterion for topology preservation is obtained using the only conditions *i*), *ii*) and *iii*) of Definition 10.5. In other words, we will prove the following theorem which states an equivalent definition to Definition 10.5. In this chapter, $(n, \overline{n}) \in \{(26, 6), (6, 26)\}$.

Theorem 16 *Let $X \subset \mathbb{Z}^3$ and $x \in X$. The voxel x is $n-$simple for $n \in \{6, 26\}$ if :*

i) X and $X \setminus \{x\}$ have the same number of $n-$connected components.

ii) $\overline{X}$ and $\overline{X} \cup \{x\}$ have the same number of $\overline{n}-$connected components.

iii) For each voxel B in $X \setminus \{x\}$, the group morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i : X \setminus \{x\} \longrightarrow X$ is an isomorphism.

Since voxels which satisfy Definition 10.5 obviously satisfy the three conditions of Theorem 16, we only have to prove that voxels which satisfy the conditions *i*), *ii*) and *iii*) of Theorem 16 do satisfy Definition 10.5. Then, we will first prove that the three conditions of Theorem 16 together imply the local characterization of simple voxels given by Proposition 10.1 (Section 10.2) and then show that this characterization itself implies that the four conditions of Definition 10.5 are satisfied.

In the sequel of this chapter, x is a voxel of X which is a subset of $\mathbb{Z}^3$ and $(n, \overline{n}) \in \{(6, 26), (26, 6)\}$.

12.1 Local characterization of the new definition

The purpose of this section is to prove the following proposition which state that conditions i), ii) and iii) of Theorem 16 imply that the voxel x satisfies a local characterization of simple voxels which involves the topological numbers (see Section 10.2).

Proposition 12.1 *If x satisfies the conditions of Theorem 16 then $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$.*

In order to prove this proposition, we introduce several other propositions and lemmas. The proof of the following Proposition is adapted from [11] to the formalism used here which involves the digital fundamental group.

Proposition 12.2 *If $T_n(x, X) \geq 2$, then either an $n-$connected component of X is created by deletion of x, or there exists a voxel $B \in X \setminus \{x\}$ such the the morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion of $X \setminus \{x\}$ in X is not onto.*

The proof of Proposition 12.2 will use the following number ν (see Section 10.2 for the definition of the geodesic neighborhood $G_n(x, X)$).

Definition 12.1 *Let C be an $n-$connected component of $G_n(x, X)$ and let $\alpha = (\alpha_i)_{i=0,\ldots,l}$ be a closed $n-$path in X. We say that the $n-$path α goes from C to x at subscript i if $\alpha_i \in C$ and $\alpha_{i+1} = x$; and we say that α goes from x to C at subscript i if $\alpha_i = x$ and $\alpha_{i+1} \in C$. Then, we define $\nu_n(x, \alpha, C)$ as the number of times α goes from C to x minus the number of time α goes from x to C.*

Lemma 12.3 *Let C be an $n-$connected component of $G_n(x, X)$ and let α and α' be two closed $n-$paths from B to B in X where $B \in X \setminus \{x\}$. If $\alpha \simeq_n \alpha'$ in X then $\nu_n(x, \alpha, C) = \nu_n(x, \alpha', C)$.*

Proof of Lemma 12.3 : It is sufficient to prove this lemma when α and α' are the same up to an elementary $n-$deformation. Then, we have $\alpha = \pi_1.\gamma.\pi_2$ and $\alpha' = \pi_1.\gamma'.\pi_2$ where γ and γ' have the same extremities and are included in a common 2×2×2 cube $\mathcal{C}$ if $(n, \overline{n}) = (26, 6)$, in a 2×2 square if $(n, \overline{n}) = (6, 26)$. It is obvious that $\nu_n(x, \alpha, C) - \nu_n(x, \alpha', C) = \nu_n(x, \gamma, C) - \nu_n(x, \gamma', C)$.

• Case $(6, 26)$: In this case, $\mathcal{C}$ is a 2×2 square. If $x \notin \mathcal{C}$ then it is clear that $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = 0$. If $x \in \mathcal{C}$ and $\mathcal{C} \cap C = \emptyset$ then $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = 0$.

Now, if $x \in \mathcal{C}$ and $\mathcal{C} \cap C \neq \emptyset$ then let a and b be the two extremities of γ and γ'.

If x has only one $6-$adjacent voxel in $X \cap \mathcal{C}$, then since $\mathcal{C} \cap C \neq \emptyset$ it follows that this voxel belongs to C. In this case, $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = 0$ if $a = b = x$ or $\{a, b\} \subset C$, $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = +1$ if $a \in C$ and $b = x$, and $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = -1$ if $a = x$ and $b \in C$.

If x has two $6-$adjacent voxels in $X \cap \mathcal{C}$ and those two voxels belong to C then $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = -1$ if $a \in C$ and $b = x$, $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = +1$ if $a = x$ and $b \in C$, $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C) = 0$ if $a = b = x$ or $\{a, b\} \in C$.

If x has two $6-$adjacent voxels in $X \cap \mathcal{C}$ and only one of these voxels, say d, belongs to C, then the remaining voxel r of $\mathcal{C}$ which is $18-$adjacent but not $6-$adjacent to x cannot be in X and so nor in C. It follows that γ and γ' are both included in $\{x, d, r\}$ and that $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C)$. Finally, in all case we have $\nu_6(x, \gamma, C) = \nu_6(x, \gamma', C)$ so that $\nu_6(x, \alpha, C) = \nu_6(x, \alpha', C)$.

• Case $(26, 6)$: If $x \notin \mathcal{C}$ then it is clear that $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C) = 0$. If $x \in \mathcal{C}$ and $\mathcal{C} \cap C = \emptyset$ then $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C) = 0$. Now, if $x \in \mathcal{C}$ and $\mathcal{C} \cap C \neq \emptyset$ then $(\mathcal{C} \cap X) \subset C$ so γ,γ' are contained in $C \cup \{x\}$. Let a and b be the two extremities of γ and γ'.

If $a = b = x$ then $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C) = 0$. If $a = x$ and $b \in C$ then $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C) = -1$. If $a \in C$ and $b = x$ then $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C) = +1$. If $\{a, b\} \subset C$ then $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C) = 0$. Finally, in all case we have $\nu_{26}(x, \gamma, C) = \nu_{26}(x, \gamma', C)$ so that $\nu_{26}(x, \alpha, C) = \nu_{26}(x, \alpha', C)$. □

Proof of Proposition 12.2 : Let C_1 and C_2 be two $n-$connected components of $G_n(x, X)$ which are $n-$adjacent to x. If C_1 and C_2 are not $n-$connected in $X \setminus \{x\}$, since they are $n-$connected in X then a new $n-$connected component is created by deletion of x.

Now, suppose that C_1 and C_2 are connected in $X \setminus \{x\}$. Let a and b be two voxels of X which are $n-$adjacent to x and such that $a \in C_1$ and $b \in C_2$. Then, there exists an $n-$path π from a to b in $X \setminus \{x\}$. Now, let π' be the closed $n-$path $(a).\pi.(b, x, a)$. It is clear that $\nu_n(x, \pi', C_1) = +1$ since $x \notin \pi^*$. Suppose that there exists in $A^a_n(X \setminus \{x\})$ a closed $n-$path α such that $i_*([\alpha]_{\Pi^n_1(X \setminus \{x\}, a)}) = [\alpha]_{\Pi^n_1(X,a)} = [\pi']_{\Pi^n_1(X,a)}$. Then, α would be $n-$homotopic to π' in X, but since $\alpha \in A_n^a(X \setminus \{x\})$ it follows that $\nu_n(x, \alpha, C_1) = 0$ whereas $\nu_n(x, \pi', C_1) = +1$ from the very construction of the path π'. From Lemma 12.3 it follows that α cannot be $n-$homotopic to π' and then the morphism $i_* : \Pi_1^n(X \setminus \{x\}, a) \longrightarrow \Pi_1^n(X, a)$ induced by the inclusion of $X \setminus \{x\}$ in X is not onto. $\square$

Proposition 12.4 *If $T_{\overline{n}}(x, \overline{X}) = 0$ then $\overline{X}$ has one $\overline{n}-$connected less than $\overline{X} \cup \{x\}$.*

Proof : If $T_{\overline{n}}(x, \overline{X}) = 0$, then no voxel of $\overline{X}$ is $\overline{n}-$adjacent to x so that $\{x\}$ is an $\overline{n}-$connected component of $\overline{X \setminus \{x\}}$. $\square$

Now, using the linking number defined in Chapter 11 we will be able to prove the following proposition.

Proposition 12.5 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ then two $\overline{n}-$connected component of $\overline{X}$ are merged by deletion of x or there exists a voxel $B \in X \setminus \{x\}$ such that the morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion of $X \setminus \{x\}$ in X is not one to one.*

The main idea of this part is to use the linking number in order to prove Proposition 12.5. Indeed, until this work and the possible use of the linking number, one could prove that when $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ and no $\overline{n}-$connected component of $\overline{X}$ are merged by deletion of x then there exists a voxel $V \in \overline{X}$ such that the morphism ${i_*}' : \Pi_1^{\overline{n}}(\overline{X}, V) \longrightarrow \Pi_1^{\overline{n}}(\overline{X} \cup \{x\}, V)$ induced by the inclusion of $\overline{X}$ in $\overline{X} \cup \{x\}$ is not onto. In other words, "a hole is created in $\overline{X} \cup \{x\}$" by deletion of x in X. Such a proof is similar to the proof of Proposition 12.2. Here, we show that in this case "a hole is created in $X \setminus \{x\}$" too. More precisely, we prove that there exists a voxel $B \in X \setminus \{x\}$ such that the morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion of $X \setminus \{x\}$ in X is not one to one.

Before proving Proposition 12.5, we must state several lemmas which use the following definitions.

Definition 12.2 (6−extremity voxel) *Let x be a voxel of $Z \subset \mathbb{Z}^3$, then x is said to be* a 6−extremity voxel of Z *if x has exactly one $6-$neighbor in Z.*

Definition 12.3 (set $K_6(y, X, C)$) *Let $y \in X$ such that $T_6(y, X) = 1$ and $T_{26}(y, \overline{X}) \geq 2$. Let $A = G_6(y, X)$, which is $6-$connected, and C be a $26-$connected component of*

$G_{26}(y, \overline{X})$. *Then,* $K_6^0(y, X, C)$ *is the set of voxels of* A *which are* $26-$*adjacent to a voxel of* C*. We define* $K_6(y, X, C)$ *as the set obtained after recursive deletions of* $6-$*extremity voxels in* K_6^0.

Definition 12.4 (26−**Bold voxel)** *Let* y *be a voxel of* X*, then* y *is a* 26−bold voxel in X *if all the voxels of* X *which are* $26-$*adjacent to* y *are included in a common* $2{\times}2{\times}2$ *cube.*

Definition 12.5 (set $K_{26}(y, X, C)$**)** *Let* $y \in X$ *such that* $T_{26}(y, X) = 1$ *and* $T_6(y, \overline{X}) \geq 2$*. Let* $A = G_{26}(y, X)$*, which is* $26-$*connected, and* C *be a* $6-$*connected component of* $G_6(y, \overline{X})$*. Then,* $K_{26}^0(y, X, C)$ *is the set of voxels of* A *which are* $6-$*adjacent to a voxel of* C*. We define* $K_{26}(y, X, C)$ *as the set obtained after iterative deletions of* $26-$*bold voxels in* K_{26}^0.

Lemma 12.6 *If* $T_n(x, X) = 1$ *and* $T_{\overline{n}}(x, \overline{X}) \geq 2$*, then there exists an* $\overline{n}-$*connected component* C *of* $G_{\overline{n}}(x, \overline{X})$ *such that* $K_n(x, X, C)$ *is a simple closed* $n-$*curve.*

Proof : In order to prove this Lemma, we have investigated using a computer all the 2^{26} possible configurations of $N_{26}(x)$. For each configuration such that $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ (there are 34653792 such configurations if $(n, \overline{n}) = (26, 6)$ and 4398983 for the case $(n, \overline{n}) = (6, 26)$), we have computed the different $\overline{n}-$connected components C_i of $G_{\overline{n}}(x, \overline{X})$ and checked that for at least one of them, the set $K_n(x, X, C_i)$, which can be computed following Definition 12.3 or Definition 12.5, was a simple closed $n-$curve. The source of this program in C programming language can be found in Appendix A. □

Lemma 12.7 *Let* $x \in X$ *such that* $T_n(x, X) = 1$ *and* $T_{\overline{n}}(x, \overline{X}) \geq 2$ *and let* $A = G_n(x, X)$*. Then there exists a parameterized simple closed* $n-$*curve* c *in* A *and a closed* $\overline{n}-$*path* $\beta = (a).\beta'.(b, x, a)$ *such that :*

- $\beta^* \subset \overline{N_{26}(x) \cap X}$,
- *a and b are the only voxels of* β' *in* $N_{26}(x)$,
- *If* $(n, \overline{n}) = (6, 26)$ *then* $L_{c,\beta} = \pm 1$ *and if* $(n, \overline{n}) = (26, 6)$ *then* $L_{\beta,c} = \pm 1$.

See Definition 11.5 for the definition of the linking number $L_{\beta,c}$ *or* $L_{c,\beta}$.

Proof of Lemma 12.7 in the case $(n, \overline{n}) = (6, 26)$ **:**

From Lemma 12.6, if $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ one can find a simple closed 6−curve $c = K_6(x, X, C)$ in $G_6(x, X)$ for some C. Furthermore, from the very definition of the set $K_6(x, X, C)$, each voxel of this curve is 26−adjacent to the 26−connected component C of $G_{26}(x, \overline{X})$. In Figure 12.1, we have depicted up to rotations and symmetries all the possible simple closed 6−curves in the 26−neighborhood of a voxel x. We will investigate each kind of curve and show that for each one a convenient simple closed 6−curve can be found in $G_6(x, X)$ together with a closed 26−path β in $\overline{X} \cup \{x\}$ which satisfy the properties of Lemma 12.7.

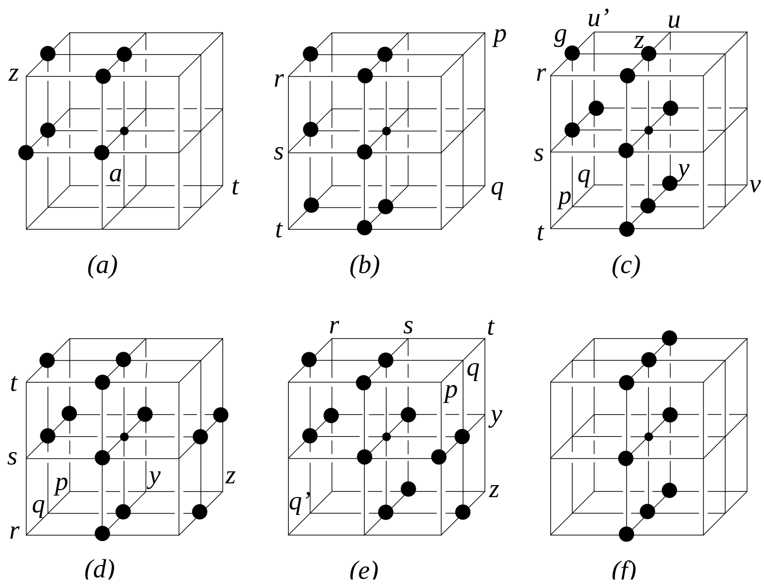


Figure 12.1: The possible simple closed 6−curves (in black points) in $N_{18}(x)$ up to rotations and symmetries.

□

Case of Figure 12.1(a)

From the definition of $K_6(x, X, C)$, each point of c must be 26−adjacent to C. Then, two cases may occur : either C is constituted by the unique voxel z or not. If C is reduced to the voxel z, then since $T_{\overline{n}}(x, \overline{X}) \geq 2$, at least one of the remaining "not black" voxels must belong to some connected component of $G_{26}(x, \overline{X}) = N_{26}(x) \cap X$ different from $C = \{z\}$. Let u be such a voxel, then it is clear that u and z can be connected by a 26−path β' in $\overline{N_{26}(x) \cap X}$ such that $L_{c,(u).\beta'.(z,x,u)} = \pm 1$ as depicted in Figure 12.2 where c is the set of voxels of Figure 12.1(a). In this figure, it is clear that the only couple of subscripts of c and $\beta = (u).\beta'.(z, x, u)$ which have a contribution (see Definition 11.4) different from 0 is the couple corresponding to the voxel x in β and a in c. Now, from the definition of this contribution, we have $L_{c,\beta} = \pm 1$.

Now, if $z \notin C$ and $z \notin X$ then z constitutes a $26-$connected component of $G_{26}(x, \overline{X})$ and in this case it can be linked to any voxel of C by a path β' such that the path $\beta = (u).\beta'.(z, x, u)$ satisfies the properties of Lemma 12.7 with the simple closed $6-$curve c constituted by the black voxels of Figure 12.1(a).

Finally the case when $z \notin C$ and $z \in X$ remains. In this case, since any point of $K_6(x, X, C)$ must be $26-$adjacent to C, and from the fact that $G_{26}(x, \overline{X})$ must have two connected components, one of the connected components must be reduced to the point t of Figure 12.1(a). Now, it follows that all the points of $N_{26}(x) \cap N_{18}(t)$ must belong to X. Otherwise, it clear that t would be $26-$adjacent to C; indeed for any point v of $N_{26}(x) \cap N_{18}(t)$ it is possible to find a point c^i of the $6-$path c such that any point of $N_{26}(x) \setminus (c^* \cup \{z\})$ which is $26-$adjacent to c^i is also $26-$adjacent to v. We obtain the configuration depicted in Figure 12.3. Now, let c' be the simple closed $6-$curve constituted by the $18-$neighbors of t, this curve is included in C since C is connected and all its points belong to $G_6(x, X)$. Furthermore, some of the points represented in dotted lines in Figure 12.3 must not be in X (otherwise, $T_{26}(x, X)$ would be equal to 1). Let u be one of these points; similarly with the previous case, one can construct a $26-$path β' between t and u such that the path $\beta = (t).\beta'.(u, x, t)$ satisfies the properties of Lemma 12.7 with c'.

Case of Figure 12.1(b)

Let c_1 be the simple closed $6-$curve constituted by the black points of Figure 12.1(b). If the component C is a subset of the set of points $\{r, s, t\}$ then the point s must belong to C since C is $26-$connected and any point of c_1 must be $26-$adjacent to C. Now, there must exist at least another $26-$connected component of white voxels (of $\overline{X}$) in $N_{26}(x) \setminus (\{r, s, t\} \cup c_1^*)$. Let u be one point of this component, then there exists an $\overline{n}-$path β' between s and u such that the path $\beta = (u).\beta'.(s, x, u)$ satisfy the properties of Lemma 12.7 with the simple closed $6-$curve c_1.

Now, suppose that $C \cap \{r, s, t\} = \emptyset$. Same considerations allow us to find such paths β and c also when the point s belongs to $\overline{X}$, or when $s \in X$ but only one of the points r and t belong to $\overline{X}$. In the case when $s \in X$ and $\{r, t\} \subset \overline{X}$ (see Figure 12.4) a path β' linking the two voxels r and t is such that the path $\beta = (r).\beta'.(t, x, r)$ satisfies the properties of Lemma 12.7 with a simple closed $6-$curve c_2 constituted by the voxels of $N_{26}(x) \cap X$ which are $18-$adjacent to r.

The case when $\{r, s, t\} \subset X$ remains. In this case, the same argument as used in the case of Figure 12.1(a) for the point t implies that the $26-$connected component of $G_{26}(x, \overline{X})$ distinct from C is reduced to p or to q, say q as in Figure 12.1(b). It follows that the voxels of $N_{26}(x) \cap N_{18}(q)$ must belong to X, particularly these voxels constitute a simple closed $6-$curve c_3. Now, let u be any voxel of the component C, u and q can be linked by a path β' such that the path $\beta = (q).\beta'.(u, x, q)$ and the curve c_3 satisfy the properties of Lemma 12.7.

Case of Figure 12.1(c)

Let c be the set of black points of Figure 12.1(c).

First, we prove that $\{p, q, r, s, t\} \cap \overline{X} \neq \emptyset$. Indeed, suppose that $\{p, q, r, s, t\} \subset X$. Then, since $T_{26}(x, \overline{X}) \geq 2$, there must exist at least two $26-$connected component in $B = (\overline{c^*} \cap N_{26}(x)) \setminus \{p, q, r, s, t\}$. Since the the point g must be $26-$adjacent to C, either the point u or the point u' must belong to C. If $u' \in C$ and $u \notin C$, then the point y (and few other ones) could not be $26-$adjacent to C since $C = \{u'\}$ in this case. It follows that u must belong to C. Now, the other $26-$connected of $G_{26}(x, \overline{X})$ must be reduced to the point v (for any remaining point $v' \in B$ different from v, it is possible to find a point w of c such that any voxel of B which is $26-$adjacent to w is either equal or $26-$adjacent to v' so that this point v' would belong to C if it belongs to $\overline{X}$). Furthermore, the points of $N_{26}(x) \cap N_{18}(v)$ must belong to X as depicted in Figure 12.5 so that the voxel y of Figure 12.5 which belongs to c is not $26-$adjacent to C. Then, $\{p, q, r, s, t\} \cap \overline{X} \neq \emptyset$.

If $s \in X$ and $r \in \overline{X}$ then C cannot be reduced to r neither included in $\{t, p, q\}$ since any black point must be $26-$adjacent to C. It follows that $\{u, u'\} \cap C \neq \emptyset$ (g must be $26-$adjacent to C). A path β' connecting r to either u or u' is such that the path $\beta = (r).\beta'.(u, x, r)$ or $\beta = (r).\beta'.(u', x, r)$ satisfies the properties of Lemma 12.7 together with c.

If $s \in X$ and $r \in X$, then either u or u' belongs to C and a path β' connecting u or u' to a point of $\{t, p, q\}$ which must belong to $\overline{X}$, is such that we may easily construct a closed path β which satisfies the properties of Lemma 12.7 together with c.

If $s \in \overline{X}$ and $p \in X$, C cannot contain s since y must be $26-$adjacent to C. It follows that C cannot contain any point of $\{r, s, t, p, q\}$ so that either u or u' must belong to C.

Finally, a path β' linking s to either u or u' is such that the path $\beta = (s).\beta'.(u, x, s)$ or $\beta = (s).\beta'.(u', x, s)$ satisfies the properties of Lemma 12.7 together with c.

If $s \in \overline{X}$ and $p \in \overline{X}$, since $T_{26}(x, \overline{X}) \geq 2$, there must exist a voxel w of $\overline{X}$ in $(\overline{c^*} \cap N_{26}(x)) \setminus \{p, q, r, s, t\}$. Then, a convenient 26−path β' linking s to w can be found such that the path $\beta = (s).\beta'.(w, x, s)$ satisfies the properties of Lemma 12.7 together with c.

Case of Figure 12.1(d)

Let c be the set of black point of Figure 12.1(d).

We prove that one point of $\{p, q, r, s, t, y, z\}$ must belong to $\overline{X}$. Indeed, otherwise the existence of a connected component of $G_{26}(x, \overline{X})$ distinct from C in $\overline{c^* \cup \{p, q, r, s, t, y, z\}} \cap N_{26}(x)$ would contradict the fact that any point of c is 26−adjacent to C.

Now, let u be a point of $\{p, q, r, s, t, y, z\} \cap \overline{X}$. If $u \in C$ then we can easily check that any of the points in $\{p, q, r, s, t, y, z\}$ either belong to X or to C. Then, since $T_{26}(x, \overline{X}) \geq 2$, there must exist a point v of $\overline{X}$ in $\overline{c^* \cup \{p, q, r, s, t, y, z\}} \cap N_{26}(x)$. Now, a 26−path β' linking u to v is such that the path $\beta = (u).\beta'.(v, x, u)$ does satisfy the properties of Lemma 12.7 together with c.

Case of Figure 12.1(e)

In this case, one of the points q or q' must belong to $\overline{X}$. Indeed, otherwise some 26−connected component C of $G_{26}(x, \overline{X})$ such that any voxel of the curve c is adjacent to C could not exist. We can suppose without loss of generality that q belongs to $\overline{X}$ (the case when $q' \in \overline{X}$ is symmetric).

Now, if q does not belong to C then C cannot contain any point of $\{r, s, t, q, p, y, z\}$ (because all the points of c must be 26−adjacent to C). So there must exist a point u of C in $\overline{c^* \cup \{r, s, t, p, q, y, z\}} \cap N_{26}(x)$. Then, a path β' between u and q is such that the path $\beta = (u).\beta'.(q, x, u)$ does satisfy the properties of Lemma 12.7. If q belongs to C, then any point of $\{r, s, t, p, y, z\}$ must either belong to X or to C (from the fact that any black point must be 26−adjacent to C). Since $T_{26}(x, \overline{X}) \geq 2$ there must exist some points of $\overline{X}$ in $\overline{c^* \cup \{r, s, t, p, y, z\}} \cap N_{26}(x)$. Let u be any such point, then a path β' linking q to u is such that the path $\beta = (q).\beta'.(u, x, q)$ does satisfy with c the properties of Lemma 12.7.

Case of Figure 12.1(f)

Let c be the set of black points in this figure. If we suppose that one of the "sides" of c in $N_{26}(x)$ does not contain any point of $\overline{X}$, then $N_{26}(x) \cap X$ contains at least the black points depicted in Figure 12.6. But in this case, any point of c must be $26-$adjacent to a component C of $G_{26}(x, \overline{X})$, then any voxel depicted with dotted circles which would belong to $\overline{X}$ would also be connected to C. This contradict the fact that $T_{26}(x, \overline{X}) \geq 2$. It follows that both sides of c in Figure 12.1(f) must contain a voxel of $\overline{X}$ and it is clear that two voxels u and v of $\overline{X}$ in each side can be linked by a path β' such that $\beta = (u).\beta'.(v, x, u)$ and the curve c satisfy the properties of Lemma 12.7.

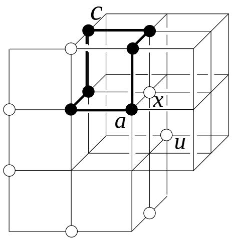


Figure 12.2: A $6-$curve c and a $26-$path β such that $L_{c,\beta} = \pm 1$.

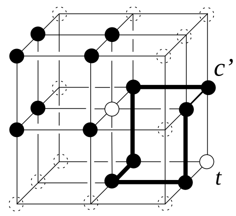


Figure 12.3:

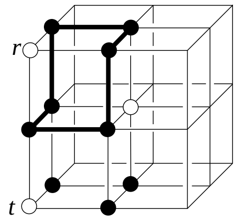


Figure 12.4:

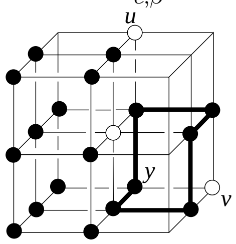


Figure 12.5:

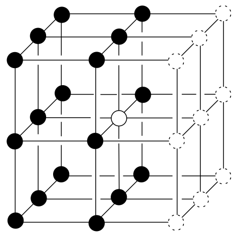


Figure 12.6:

Proof of Lemma 12.7 in the case $(26, 6)$ **:**

From Lemma 12.6, if $T_{26}(x, X) = 1$ and $T_6(x, \overline{X}) \geq 2$ one can find a simple closed $26-$curve $c = K_{26}(x, X, C)$ in $G_{26}(x, X)$ for some C. In fact, from the very definition of $K_{26}(x, X, C)$, it is clear that the curve $K_{26}(x, X, C)$ is included in $N_{18}(x)$. Indeed, c cannot contain any voxel of $N_{26}(x) \setminus N_{18}(x)$ since obviously such a voxel would be a bold $26-$voxel which cannot occur in $K_{26}(x, X, C)$.

Furthermore, from the very definition of the set $K_{26}(x, X, C)$, each point of this curve is $6-$adjacent to some $6-$connected component of $G_6(x, \overline{X})$. In Figure 12.9, we have depicted up to rotations and symmetries all the possible simple closed $26-$curve c in the $18-$neighborhood of a point x. We will investigate each kind of curve and show that for each one a convenient simple closed $26-$curve can be found in $G_{26}(x, X)$ together with a closed $6-$path β in $\overline{X} \cup \{x\}$ which satisfy the properties of Lemma 12.7.

Case of Figure 12.9(a)

Let c be the simple closed $26-$curve constituted by the black points of Figure 12.9(a), then the points u and v must both belong two $\overline{X}$. Indeed, there must exist at least two connected component in $G_6(x, \overline{X})$ and so at least two points of $\overline{X}$ must be $6-$adjacent to x. Now, the two voxels u and v can be connected by a $6-$path β' such that c with $\beta = (u).\beta'.(v, x, u)$ satisfy the properties of Lemma 12.7.

Case of Figure 12.9(b)

If u belongs to $\overline{X}$, then since $T_6(x, \overline{X}) \geq 2$, at least one of the points of $\{p, q, r, s, t\}$ must belong to $\overline{X}$. A path β' from such a point $\omega \in \{p, q, r, s, t\}$ to the point u is such that the path $\beta = (\omega).\beta'.(u, x, \omega)$ satisfies the properties of Lemma 12.7 together with a curve c constituted by the black points of Figure 12.9(b).

If $u \in X$, then since each point of the curve must be $6-$adjacent to C, the component C contains the points p, q, r and s; and since $T_6(x, \overline{X}) \geq 2$ the point t must belong to $\overline{X}$ and must not be connected in $G_6(x, \overline{X})$ to the set $\{p, q, r, s\}$. It follows that the points of $N_{26}(x)$ which are $6-$adjacent to the point t must belong to X, furthermore they constitute a simple closed $26-$curve c such that c with a path $\beta = (q).\beta'.(t, x, q)$ (where β' links the voxels q and t) does satisfy the properties of Lemma 12.7.

Case of Figure 12.9(c)

Since the black point y must be $6-$adjacent to C, then either $u \in C$ or $s \in C$. If $u \in C$, then at least one point in $\{p, q, r, s, t\}$, say p, must belong to another $6-$connected component of $G_6(x, \overline{X})$. Then a $6-$path β' linking p to u can be found such that the path $\beta = (p).\beta'.(u, x, p)$ satisfies the properties of Lemma 12.7 together with the curve c constituted by the black points of Figure 12.9(c).

Now, if $s \in C$ and $u \in \overline{X}$ then a path $\beta = (s).\beta'.(u, x, s)$ where β' is a path from s to u is convenient, still with the simple closed $26-$curve c constituted by the black points of Figure 12.9(c).

Finally, if $s \in C$ and $u \in X$ and from the fact that any black point of the simple closed $26-$curve of Figure 12.9(c) must be $6-$adjacent to C, we deduce that C contains the points p, s and r. From the existence of another $6-$connected component in $G_6(x, \overline{X})$, the point t must belong to such a component. Since t must not be connected to the points s, r and p in $G_6(x, \overline{X})$, the points b, c and d of Figure 12.7 must belong to X. In this case, the simple closed $26-$curve c constituted by the points a, b, c and d of Figure 12.7 together with the $6-$path $\beta = (s).\beta'.(t, x, s)$ where β' links s to t, do satisfy the properties of Lemma 12.7.

Case of Figure 12.9(d)

Since y is 6−adjacent to C, either $u \in C$ or $q \in C$. If $u \in C$ then s must also belong to C since C is 6−connected and since any black point of the curve must be 6−adjacent to C, then some point ω in $\{p, q, r, t\}$ must be in $\overline{X}$ since $T_6(x, \overline{X}) \geq 2$. A path β' from such a point to the point u can be found so that the path $\beta = (\omega).\beta'.(u, x, \omega)$ satisfies the properties of Lemma 12.7 with the simple closed 26−curve c constituted by the black points of Figure 12.9(d).

The case when $u \notin C$ but $u \in \overline{X}$ is similar since q must belong to C in this case. The path β' being then considered links u and q.

Now, if $u \in X$ and so $q \in C$ it is then clear that r and p must belong to C. We also prove that in this case s must belong to $\overline{X}$. Indeed, suppose that $s \in X$, then the point t must belong to a 6−connected component of $G_6(x, \overline{X})$ distinct from C. It follows that the points of $N_6(t) \cap N_{26}(x)$ must belong to X and $N_{26}(x) \cap X$ contains at least the black points of Figure 12.8. But in this case, is is clear that the point a of Figure 12.8 which belongs to the simple closed curve depicted in Figure 12.9(d) cannot be 6−adjacent to the component C. Then, $s \in \overline{X}$. Now, if $s \in \overline{X}$, a path $\beta = (r).\beta'.(s, x, r)$ where β' links r to s, together with the curve c constituted by the black points of Figure 12.9(d), will satisfy the properties of Lemma 12.7.

Case of Figure 12.9(e)

In this case, let c be the simple closed 26−curve constituted by the black points of Figure 12.9(e).

Since y is 6−adjacent to C then either $u \in C$ or $v \in C$. If $u \in C$ then q must belong to C too and then one point in $\{v, p, r\}$, say v, must belong to $\overline{X}$ since $T_6(x, \overline{X}) \geq 2$. A 6−path β' linking v to u can be found such that the path $\beta = (v).\beta'.(u, x, v)$ satisfies the properties of Lemma 12.7 together with the curve c. If $v \in C$ then r and p must belong to C and either u or q, say q, must belong to another 6−connected component of $G_6(x, \overline{X})$ so belongs to $\overline{X}$. Then, a path $\beta = (v).\beta'.(q, x, v)$ v, where β' links v to q, will still satisfy the properties of Lemma 12.7.

Case of Figure 12.9(f)

In this case, let c be the simple closed 26−curve constituted by the black points of Figure 12.9(f).

Since y is $6-$adjacent to C then either $u \in C$ or $v \in C$.

If $u \in C$ then q and t must belong to C too since each point of c is $6-$adjacent to a point of C which must be $6-$connected. It follows that at least one point in $\{r, v, p\}$ must belong to another $6-$connected component of $G_6(x, \overline{X})$ and so belongs to $\overline{X}$. Finally, a $6-$path linking a point of $\{r, v, p\}$ to u will be such that one can find a path β which satisfies the properties of Lemma 12.7 together with the curve c. The case when $v \in C$ is similar.

Case of Figure 12.9(g)

Let c be the simple closed $26-$curve constituted by the black points of Figure 12.9(g). If $u \in C$ then q must belong to C and then either v or r belongs to a $6-$connected component of $G_6(x, \overline{X})$ and so belongs to $\overline{X}$. Then, a path β' linking v or r to u is such that the path $\beta = (v).\beta'.(u, x, v)$ or $\beta = (r).\beta'.(u, x, r)$ satisfies the properties of Lemma 12.7 with the curve c. The case when $v \in C$ is similar.

Cases of Figures 12.9(h), 12.9(i), 12.9(j) and 12.9(h)

These cases are similar to the previous ones.

Case of Figure 12.9(l)

The point y must be $6-$adjacent to C so that either $u \in C$ or $v \in C$. Now, we prove that u cannot belong to X. Indeed, if $v \in C$ and $u \in X$, then the point r must belong to C too since y' must be $6-$adjacent to C. Then, since C is $6-$connected, there must exist a $6-$path from r to v in $G_6(x, \overline{X}) \subset N_{18}(x)$. It is clear that such a path must contain the point q and this contradict the fact that $T_6(x, \overline{X}) \geq 2$. Finally, $u \in \overline{X}$.

If $u \in C$ then one point of $\{v, q, r\}$ must belong to a $6-$connected component of $G_6(x, \overline{X})$ distinct from C and a $6-$path β' from one of these points, say v, to u can be found such that the path $\beta = (v).\beta'.(u, x, v)$ satisfies the properties of Lemma 12.7 with the curve c constituted by the black points of Figure 12.9(l).

The case when $v \in C$ and $u \in \overline{X}$ is similar.

Case of Figure 12.9(m) and Figure 12.9(n)

These cases are similar to the previous one and use the same arguments.

□

Lemma 12.8 *Let x be a point of X such that $T_n(x, X) = 1$. Then, any closed $n-$path c in $G_n(x, X)$ is $n-$reducible in X.*

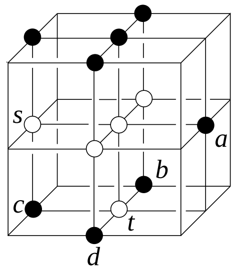


Figure 12.7:

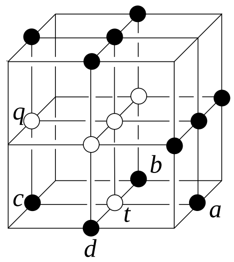


Figure 12.8:

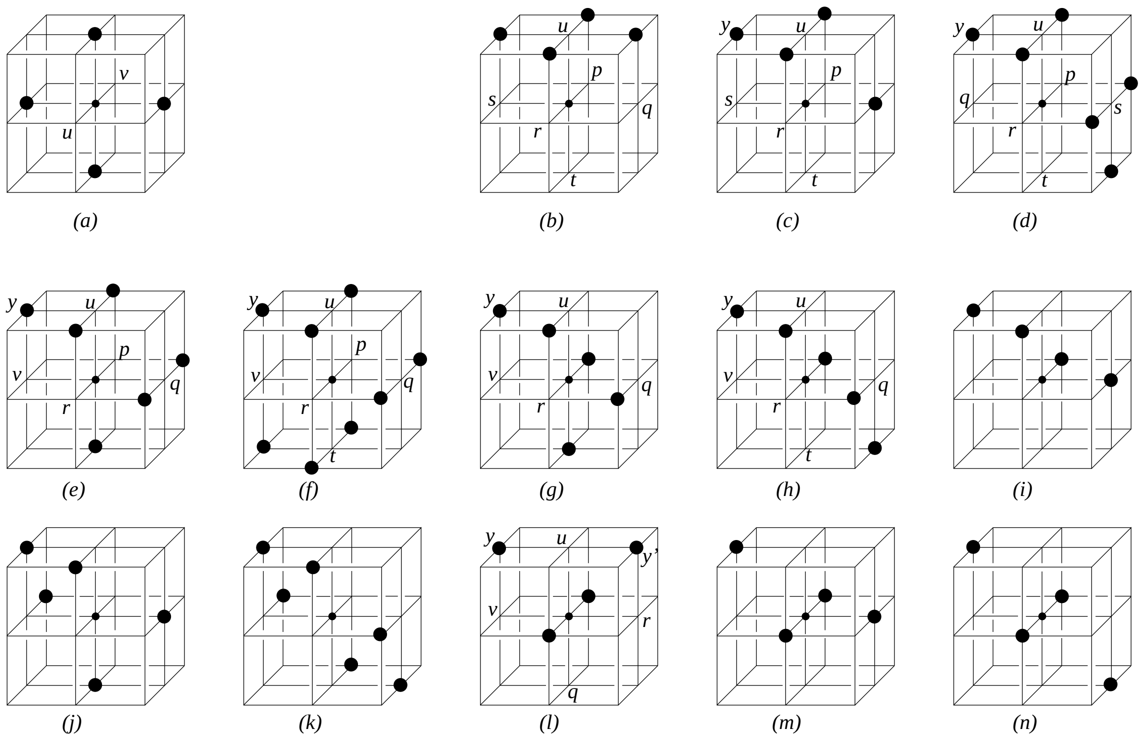


Figure 12.9: The possible simple closed 26−curves with a length greater than three in $N_{26}(x)$ up to rotations and symmetries.

Proof : Let $c = (c_0, \ldots, c_p)$ with $c_0 = c_p$. If $(n, \overline{n}) = (26, 6)$, then let c' be the closed path obtained after insertion of the point x in c between any two consecutive points of c. It is clear that $c \simeq_{26} c'$ in X since for any two consecutive points of c, x belongs to some 2×2×2 cube which contains these two points. Now, c' is of the following form : $c' = (c_0, x, c_1, x, \ldots, x, c_n)$. In c', each sequence of the form (x, c_i, x) can be reduced to (x) by an elementary 26−deformation. It follows that $c \simeq_{26} c' \simeq_{26} (c_0, x, c_n) \simeq_{26} (c_0, c_n)$. If $(n, \overline{n}) = (6, 26)$, we first observe that any closed 6−path in $N_{18}(x)$ can be deformed in X into a path which only contains multiple occurrences of the point x and 6−neighbors of x in X. Indeed, any point z of c which belongs to $N_{18} \setminus N_6(x)$ occurs in a sub-sequence (u, z, v) (note that c can also be made of a single voxel of $N_{18}(x) \cap X$). Then, u and v are 6−neighbors of x and the points u, z, v and x are included in a 2×2 square. It follows that the sequence (u, z, v) can be replaced by the sequence (u, x, v) in c by an elementary 6−deformation. By repeating this deformation for any such point z in c will lead to a path c' such that $c'^* \subset \{x\} \cup (N_6(x) \cap X)$ and it is then immediate that $c' \simeq_6 (c_0, c_p)$ in X. □

Proof of Proposition 12.5 : Let x be a point of X such that $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$. Let c, β' and β be the paths of Lemma 12.7 and a and b be the extremity voxels of β' which are the only two points of β' in $N_{26}(x)$ and which are $\overline{n}$−adjacent to x. If a and b are not $\overline{n}$−connected in $\overline{X}$ then it is clear that they are $\overline{n}$−connected in $\overline{X} \cup \{x\}$ so that two $\overline{n}$−connected components of $\overline{X}$ are merged by deletion of x from X. If a and b are connected by an $\overline{n}$−path α in $\overline{X}$. Then, it is obvious that the two $\overline{n}$−paths β' and α are $\overline{n}$−homotopic with fixed extremities in $\overline{(N_{26}(x) \cap X) \cup \{x\}}$. It follows that β is $\overline{n}$−homotopic to the path $\alpha' = (a).\alpha.(b, x, a)$ in $\overline{(N_{26}(x) \cap X)}$. Since $\overline{(N_{26}(x) \cap X)} \subset \overline{c^*}$ and from Theorem 15 then $L_{c,\beta} = L_{c,\alpha'} = \pm 1$.

From Theorem 14, it follows that the path c is not n−reducible in $\overline{\alpha'^*}$ and since $\alpha'^* \subset \overline{X} \cup \{x\}$ then $X \setminus \{x\} \subset \overline{\alpha'^*}$ so that *a fortiori* α' cannot be n−homotopic to a trivial path in $X \setminus \{x\}$. Formally, if B is the voxel of $X \setminus \{x\}$ such that c is a closed n−path from B to B, we have $[c]_{\Pi_1^n(X \setminus \{x\}, B)} \neq [1]_{\Pi_1^n(X \setminus \{x\}, B)}$.

Now, from Lemma 12.8, $c \simeq_n (B, B)$ in X so that $i_*([c]_{\Pi_1^n(X \setminus \{x\}, B)}) = [c]_{\Pi_1^n(X, B)} = [1]_{\Pi_1^n(X, B)} = i_*([1]_{\Pi_1^n(X \setminus \{x\}, B)})$ so i_* is not one to one. □

Proof of Proposition 12.1 : Suppose that properties $i)$, $ii)$ and $iii)$ of Definition 10.5 are satisfied.

From Proposition 12.2 we deduce that if i_* is onto for any voxel B in $X \setminus \{x\}$, and no $n-$connected component of X is created by deletion of x then $T_n(x, X) < 2$. Furthermore, if no connected component of X is removed then $T_n(x, X) \neq 0$ (indeed, $T_n(x, X) = 0$ means that x constitutes an $n-$connected component of X since no other point of X is $n-$adjacent to x). Finally, $T_n(x, X) = 1$.

From Proposition 12.5 we deduce that if i_* is one to one for any voxel $B \in X \setminus \{x\}$, and no $\overline{n}-$connected components of $\overline{X}$ are merged by addition of x in $\overline{X}$ then $T_{\overline{n}}(x, \overline{X}) < 2$. Furthermore, if no connected component of $\overline{X}$ is created then $T_{\overline{n}}(x, \overline{X}) \neq 0$ (indeed, $T_{\overline{n}}(x, \overline{X})$ means that no point of $\overline{X}$ is $\overline{n}-$adjacent to x so that x constitutes an $\overline{n}-$connected component of $\overline{X} \cup \{x\}$). Finally, $T_{\overline{n}}(x, \overline{X}) = 1$. $\square$

12.2 The local characterization implies the previous definition

In this section, we prove that the four properties of Definition 10.5 are satisfied when $T_n(x, X) = T_{\overline{n}}(x, \overline{X}) = 1$.

Proposition 12.9 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$, then conditions i), ii), iii) and iv) of Definition 10.5 are satisfied.*

In order to prove Proposition 12.9 we will state several propositions.

Proposition 12.10 *If the set X has more $n-$connected components then $X \setminus \{x\}$ then $T_n(x, X) = 0$.*

Proof : If X has more $n-$connected components then $X \setminus \{x\}$ then some connected component of X is removed by deletion of x, then no other point of X can belong to this component. It follows that x has no $n-$neighbor in X and then $T_n(x, X) = 0$. $\square$

Proposition 12.11 *If the set $X \setminus \{x\}$ has more $n-$connected components than X then $T_n(x, X) \geq 2$.*

Proof : If $X \setminus \{x\}$ has more $n-$connected components than X, some connected component of X has been created by deletion of x. In other words, there exist two voxels

a and b in X such that a and b are connected in X but not in $X \setminus \{x\}$. It follows that every $n-$path between a and b in X contains the voxel x. Then $T_n(x, X)$ cannot be equal to zero since in this case no path between a and b in X could contain x. Now, suppose that $T_n(x, X) = 1$. In this case, for any $n-$path c between a and b in X, one can find a path c' from a to b in $X \setminus \{x\}$. Indeed, for any sequence of the form (y, x, z) in c, the points y and z both belong to $G_n(x, X)$ which is $n-$connected and so there exists an $n-$path in $X \setminus \{x\}$ between y and z. Then, any such sequence (y, x, z) in c can be replaced by an $n-$path which does not contain x so that a and b are $n-$connected in $X \setminus \{x\}$, which contradicts the definition of a and b. Finally, we have $T_n(x, X) \geq 2$. $\square$

Proposition 12.12 *If the set $\overline{X}$ has more $\overline{n}-$connected components than $\overline{X} \cup \{x\}$, then $T_{\overline{n}}(x, \overline{X}) \geq 2$.*

Proof : The proof is similar to the proof of Proposition 12.11. $\square$

Proposition 12.13 *If the set $\overline{X} \cup \{x\}$ has more $\overline{n}-$connected components than $\overline{X}$, then $T_{\overline{n}}(x, \overline{X}) = 0$.*

Proof : The proof is similar to the proof of Proposition 12.10. $\square$

Proposition 12.14 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then for all $B \in X \setminus \{x\}$ the morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion of $X \setminus \{x\}$ in X is an isomorphism.*

Corollary 12.15 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then for all $B' \in \overline{X}$ the morphism $i'_* : \Pi_1^{\overline{n}}(\overline{X}, B') \longrightarrow \Pi_1^{\overline{n}}(\overline{X} \cup \{x\}, B')$ induced by the inclusion of $\overline{X}$ in $\overline{X} \cup \{x\}$ is an isomorphism.*

Proof of Corollary 12.15 : Let $Y = \overline{X} \cup \{x\}$ and $(m, \overline{m}) = (\overline{n}, n)$. Furthermore, let B' be a voxel of $\overline{X}$. Then $T_m(x, Y) = 1$, $T_{\overline{m}}(x, \overline{Y}) = 1$ and $B' \in Y \setminus \{x\}$. From Proposition 12.14, the morphism $i_* : \Pi_1^m(Y \setminus \{x\}, B') \longrightarrow \Pi_1^m(Y, B')$ induced by the inclusion map $i : Y \setminus \{x\} \longrightarrow Y$ is an isomorphism. But, $Y \setminus \{x\} = \overline{X}$ and $Y = \overline{X} \cup \{x\}$ so i_* is the morphism induced by the inclusion of $\overline{X}$ in $\overline{X} \cup \{x\}$. $\square$

In order to prove Proposition 12.14 we will first state that i_* is onto (consequence of Lemma 12.17 below) and then sate Lemma 12.22 which will allow us two prove that i_* is one to one.

Lemma 12.16 *If $T_n(x,X) = 1$ and a and b are two points of $N_n(x) \cap X$. Then there exists a simple $n-$path γ between a and b in $G_n(x,X)$ such that $(a,x,b) \simeq_n \gamma$ in X.*

Proof : Since $G_n(x,X)$ is $n-$connected, there exists a simple $n-$path $\gamma = (y_0,\dots,y_k)$ in $G_n(x,X)$ such that $y_0 = a$ and $y_k = b$.

If $(n,\overline{n}) = (26,6)$, it is clear that the voxels $a = y_0$, x and y_1 are included in a 2×2×2 cube. Then $(a,x,b) \sim_{26} (a,y_1,x,b)$ and we can repeat this process since two consecutive voxels y_i and y_{i+1} in γ are always included in a common 2×2×2 cube with x and we obtain that $(a,x,b) \sim_{26} (a,y_1,x,b) \sim_{26} \dots \sim_{26} (a,y_1,\dots,y_{k-1},x,b)$ and finally, $(a,y_1,\dots,y_{k-1},x,b) \sim_{26} (a = y_0,y_1,\dots,y_{k-1},b = y_k)$.

If $(n,\overline{n}) = (6,26)$ then we first observe that k is necessarily even. Now, $a = y_0 \in N_6(x)\cap X$ so that $y_1 \in (N_{18}(x)\setminus N_6(x))\cap X$ and $y_2 \in N_6(x)\cap X$. Then the voxels y_0, x, y_1 and y_2 are included in a 2×2 square so that $(a = y_0,x) \sim_6 (y_0,y_1,y_2,x)$. This process can be iterated to obtain that $(a,x) \simeq_6 (y_0,\dots y_k,x)$ so that $(a,x,b) \simeq_6 (y_0,\dots,y_k,x,y_k) \sim_6 (y_0,\dots,y_k)$. □

Lemma 12.17 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) = 1$ then for all $B \in X \setminus \{x\}$ and all $n-$path c of $A_n^B(X)$, there exists a path c' in $A_n^B(X\setminus\{x\})$ such that $c \simeq_n c'$ in X.*

Proof : Let $B \in X\setminus\{x\}$ and $c = (c_0,\dots,c_q)$ be a closed $n-$path from B to B in X ($B = c_0 = c_l$). For any maximal sequence $(c_i,\dots,c_j)$ such that $c_{i-1} \neq x$, $c_{j+1} \neq x$ and $c_k = x$ for $k = i,\dots,j$ it is obvious that $c \simeq_n (c_0,\dots,c_{i-1},x,c_{j+1},\dots,c_q)$ (observe that $0 < i \leq j < l$). Now, from Lemma 12.16 and since $\{c_{i-1},c_{j+1}\} \subset N_n(x)$, then $(c_{i-1},x,c_{j+1}) \simeq_n \gamma$ in X where γ is a path from c_{i-1} to c_{j+1} in $G_n(x,X)$ so that $x \notin \gamma^*$. Finally, $c \simeq_n (c_0,\dots,c_{i-1}).\gamma.(c_{j+1},\dots,c_q)$. By repeating such an $n-$homotopic deformation for any similar maximal sequence $(c_i,\dots,c_j)$ in c, it is clear that c is $n-$homotopic in X to a closed $n-$path c' such that $x \notin c'^*$ (i.e. $c' \in A_n^B(X\setminus\{x\})$). □

Lemma 12.18 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) = 1$ then two paths π_1 and π_2 which have the same extremities and are included in $G_n(x,X)$ are $n-$homotopic with fixed extremities in $N_{26}(x) \cap X$.*

In order to prove Lemma 12.18 we will use the following lemma.

Lemma 12.19 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then any simple closed $n-$path in $G_n(x, X)$ is $n-$reducible in $N_{26}(x) \cap X$.*

Corollary 12.20 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then any closed $n-$path in $G_n(x, X)$ is $n-$reducible in $N_{26}(x) \cap X$ (i.e $G_n(x, X)$ is simply $n-$connected).*

Proof of Lemma 12.19 in the $(6, 26)$ case : In this case, it is immediate that any simple closed $6-$path in $G_6(x, X) \subset N_{18}(x) \cap X$ is a simple closed $6-$curve. In Figure 12.1 we have depicted up to rotations and symmetries all the possible simple closed $6-$curves in $N_{18}(x)$.

Case of Figure 12.1(a) :

Let c be the set of black points of Figure 12.1(a). In this case, either $z \in X$ or all points of $N_{26}(x) \setminus (c^* \cup \{z\})$ must belong to X. Indeed, the case when $z \in \overline{X}$ and some point of $N_{26}(x) \setminus (c^* \cup \{z\})$ belongs to $\overline{X}$ contradict the fact that $T_{26}(x, \overline{X}) = 1$.

Now, if $z \in X$ it is clear that c is $6-$reducible in $N_{26}(x) \cap X$, similarly when $z \notin X$ then $N_{26}(x) \setminus (c^* \cup \{z\}) \subset X$ and c is obviously $6-$reducible $N_{26}(x) \cap X$.

Case of Figure 12.1(b) : In this case, either $\{r, s, t\} \subset X$ or $N_{26}(x) \setminus (c^* \cup \{r, s, t\}) \subset X$. In both case, we can conclude as in the previous case.

Cases of Figures 12.1(c),...,(f) : are similar to the previous ones. $\square$

Lemma 12.21 *Let $x \in X$ such that $T_{26}(x, X) = 1$ and $T_6(x, \overline{X}) = 1$ and let c be the parameterization of a simple closed $26-$curve in $G_{26}(x, X)$. Then c is $26-$reducible in $G_{26}(x, X)$.*

Proof : In Figure 12.9 are depicted up to rotations and symmetries all the possible simple closed $26-$curves in $N_{26}(x)$ with a length greater than three (a simple closed $26-$curve with a length of three being obviously $26-$reducible). Now, we must investigate each of them and prove that, under the hypothesis $T_{26}(x, X) = 1$ and $T_6(x, \overline{X}) = 1$, a parameterization of each simple closed curve is $26-$reducible in $G_{26}(x, X)$. Following Lemma 2.6, if one parameterization is reducible, then any parameterization is.

Case of Figure 12.9(a)

In this case, exactly one point of $\{u, v\}$ must belong to $\overline{X}$, indeed $\{u, v\} \subset X$ contradict the fact that $T_6(x, \overline{X}) = 1$ whereas $\{u, v\} \subset X$ implies that $T_6(x, \overline{X}) = 0$. If $u \in X$ [resp. $v \in X$], it is then obvious that c is $26-$reducible in $G_{26}(x, X)$.

Case of Figure 12.9(b)

If $u \in X$ then it is clear that the curve c is $26-$reducible in $G_{26}(x, X)$. If $u \notin X$ then $\{p, q, r, s, t\} \subset X$. Indeed, otherwise $G_6(x, \overline{X})$ would not be $6-$connected. As an example, Figure 12.10 shows a sequence of elementary $26-$deformations in $G_{26}(x, X)$ which leads from c to the path reduced to its extremities when c is the parameterization of the curve which starts en ends at this latter point.

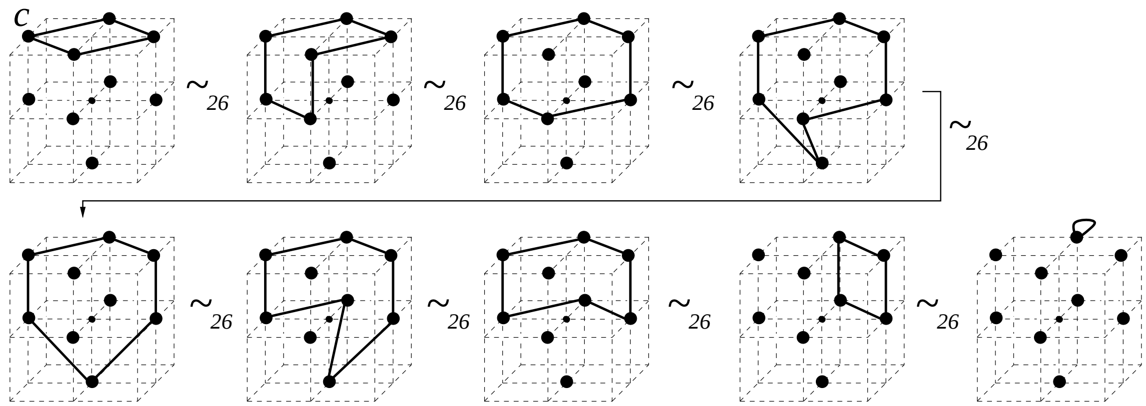


Figure 12.10: A $26-$homotopic deformation of the closed path c.

Case of Figure 12.9(c)

Since $T_6(x, \overline{X}) = 1$ we deduce that either $u \in X$ or $\{p, s, r, t\} \subset X$. In both cases, any parameterization c of the curve is $26-$reducible in $G_{26}(x, X)$.

Case of Figure 12.9(d) In this case, either $\{u, s\} \subset X$ or $\{p, q, r, t\} \subset X$ and we can conclude in both cases that any parameterization c of the simple closed curve is $26-$reducible in $G_{26}(x, X)$.

Cases of Figures 12.9(e),...,(n) In all these case, we can separate the set $N_6(x) \setminus c^*$ into two sets A and B such that either $A \subset X$ or $B \subset X$. In any case, the inclusion of one of these sets in X allows the $26-$deformation of c in $G_{26}(x, X)$ into the trivial path reduced to its extremities. $\square$

Proof of Lemma 12.19 in the $(26, 6)$ **case :** We prove this lemma by induction on the length of c. Let $c^0 = c$ and suppose that c^i is a simple closed $26-$path with a length $l(c^i)$ in $G_{26}(x, X)$ which is $26-$connected.

First, suppose that there exists in c^i three consecutive voxels which are included in a 2×2×2 cube $\mathcal{C}$. In other words, $c^i = c_1.(y, z, t).c_2$ where y, z and t belong to $\mathcal{C}$. Then, $c^i \sim_{26} c^{i+1} = c^i_1.(y, t).c^i_2$ which has a length $l(c^{i+1}) = l(c^i) - 1$.

Now, we suppose that for any sequence (y,z,t) in c^i, the two voxels y and t are not $26-$adjacent. Furthermore, suppose that there exists in c^i a voxel y such that y has more than two $26-$adjacent voxels in c^{i*}. In other words, there exists another voxel z in c^i which is neither the successor nor the predecessor of y in c^i but which is $26-$adjacent to y. Then, $c^i = c^i_1.(y).c^i_2.(z).c^i_3$ with $l(c^i_2) > 3$ (indeed, if $l(c^i_2) = 3$ then $c^i_2 = (y,u,z)$ where y is $26-$adjacent to z). We may suppose that the path c^i_2 is one of the shortest such sub-paths of c^i which can be found satisfying the $26-$adjacency property for its extremities. Then, it follows that any voxel of c^i_2 distinct from y and z has exactly two neighbors in ${c^i_2}^*$: its predecessor and its successor in c_2. Indeed, the existence of a voxel of c^i_2 which has more then two $26-$adjacent voxels in ${c^i_2}^*$ would contradict the fact that c^i_2 is a shortest sub-path of c^i whose extremities are $26-$adjacent. Furthermore, y [resp. z] has exactly two neighbors in ${c^i_2}^*$: its successor in c^i_2 and z [resp. its predecessor in c^i_2 and y]. Then, ${c^i_2}^*$ is a simple closed $n-$curve and $c^i_2.(z,y)$ is a parameterization of this curve. From Lemma 12.21, we have $c^i_2.(z,y) \simeq_{26} (y,y)$ in $G_{26}(x,X)$. On the other hand, it is obvious that $c^i \simeq_{26} c^i_1.(y).c^i_2.(z,y,z).c^i_3$ in $G_{26}(x,X)$. Finally $c^i \simeq_{26} c^i_1.(y,z).c^i_3 = c^{i+1}$ in $G_{26}(x,X)$ and c^{i+1} is a simple closed $26-$path such that $l(c^{i+1}) < l(c^i)$.

In the remaining case, any voxel of c^i has exactly two $26-$adjacent voxels in c^{i*}. Then, c^i is a parameterization of a simple closed $n-$curve and from Lemma 12.21 c^i is $n-$reducible in $G_{26}(x,X)$, i.e. $c^i \simeq_{26} c^{i+1}$ in $G_{26}(x,X)$ with $l(c^{i+1}) = 1$.

In all cases, the path c^i is $26-$homotopic to a simple closed $26-$path c^{i+1} such that $l(c^{i+1}) < l(c^i)$. By induction and since $l(c^i) \geq 1$, there must exists an integer j such that $l(c^j) = 1$ and $c^0 \simeq_{26} c^j$. □

Proof of Corollary 12.20 : If c is not simple, then there must exist a simple closed $n-$path γ from a point $y \in c^*$ to y such that $c = c_1.\gamma.c_2$. Then, from Lemma 12.19, we have $\gamma \simeq_n (y,y)$ in $G_n(x,X)$ so that $c \simeq c_1.c_2$ in $G_n(x,X)$. Now, we can iterate this process to obtain that c is $n-$homotopic to a simple closed path in $G_n(x,X)$ and finally $26-$reducible $G_n(x,X)$. □

Proof of Lemma 12.18 : Let π and π' be two $n-$paths from a voxel a to a voxel b in $G_n(x,X)$. From Corollary 12.20, the set $G_n(x,X)$ is simply $n-$connected and from Proposition 2.5 it follows that π and π' are $n-$homotopic in $G_n(x,X)$. □

Now, we will prove the following lemma which allows us to prove that the morphism i_* of Proposition 12.14 is one to one.

Lemma 12.22 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then, for any voxel $B \in X \setminus \{x\}$, two closed $n-$paths c and c' of $A_n^B(X \setminus \{x\})$ which are $n-$homotopic in X are $n-$homotopic in $X \setminus \{x\}$.*

Proof : Given a closed $n-$path c in $A_n^B(X)$, we denote by $\sigma(c)$ the $n-$path of $A_n^B(X \setminus \{x\})$ which is $n-$homotopic to c in X following the proof of Lemma 12.17. It is sufficient to prove that if c and c' are the same up to an elementary $n-$deformation in X then the two paths $\sigma(c)$ and $\sigma(c')$ are $n-$homotopic in $X \setminus \{x\}$. We suppose that $c = c_1.\gamma.c_2$ and $c' = c_1.\gamma'.c_2$ where γ and γ' are two $n-$path with the same extremities and included in a 2×2×2 cube if $(n, \overline{n}) = (26, 6)$, in a 2×2 square if $(n, \overline{n}) = (6, 26)$.

If $x \notin \gamma^* \cup \gamma'^*$ we observe that $\sigma(c) = \sigma(c_1).\gamma.\sigma(c_2)$ and $\sigma(c') = \sigma(c_1).\gamma'.\sigma(c_2)$ and then $c \sim_n c'$ in $X \setminus \{x\}$.

Now, if $x \in \gamma^* \cup \gamma'^*$ let a be the last voxel of c_1 distinct from x and let b be the first voxel of c_2 distinct from x. Then, let δ be the sub-path of c from a to b and δ' be the sub-path of c' between a and b. We denote by π_1 the sub-path of c from its first voxel to a and by π_2 the sub-path of c from b to its last voxel. Finally, we have $c = \pi_1.\delta.\pi_2$ and $c' = \pi_1.\delta'.\pi_2$. Since a and b, the two extremities of δ and δ', are distinct from x, it follows that : $\sigma(c) = \sigma(\pi_1).\sigma(\delta).\sigma(\pi_2)$ and $\sigma(c') = \sigma(\pi_1).\sigma(\delta').\sigma(\pi_2)$.

Now, since $x \in \gamma^* \cup \gamma'^*$ and since γ and γ' are $6-$paths [resp. $26-$paths] included in a 2×2 square which contains x [resp. a 2×2×2 cube], it is straightforward that γ and γ' are paths included in $G_6(x, X) \cup \{x\}$ [resp. $G_{26}(x, X)$] and from their construction so are δ and δ'. Now, from the very definition of $\sigma(\delta)$ and $\sigma(\delta')$ (see the proof of Lemma 12.17) it is straightforward that $\sigma(\delta)$ and $\sigma(\delta')$ are two $n-$paths in $G_n(x, X)$ with same extremities. From Lemma 12.18, we conclude that $\sigma(\delta) \simeq_n \sigma(\delta')$ in $N_{26}(x) \cap X \subset X \setminus \{x\}$. Finally, $\sigma(c) \simeq_n \sigma(c')$ in $X \setminus \{x\}$. □

Proof of Proposition 12.14 : Let B be a voxel of $X \setminus \{x\}$. From Lemma 12.17, for any closed path $c' \in A_n^B(X)$ (and so for any homotopic class of path $[c']_{\Pi_1^n(X,B)}$) there exists a path $c \in A_n^B(X \setminus \{x\})$ such that $c \simeq_n c'$ in X so that $i_*([c]_{\Pi_1^n(X \setminus \{x\},B)}) = [c]_{\Pi_1^n(X,B)} = [c']_{\Pi_1^n(X,B)}$. Then, the morphism i_* is onto.

Now, suppose that c_1 and c_2 are two closed paths of $A_n^B(X \setminus \{x\})$ such that $[c_1]_{\Pi_1^n(X,B)} = [c_2]_{\Pi_1^n(X,B)}$, where $[c_1]_{\Pi_1^n(X,B)} = i_*([c_1]_{\Pi_1^n(X \setminus \{x\},B)})$ and $[c_2]_{\Pi_1^n(X,B)} = i_*([c_2]_{\Pi_1^n(X \setminus \{x\},B)})$. Then, $c_1 \simeq_n c_2$ in X and from Lemma 12.22, it follows that $c_1 \simeq_n c_2$ in $X \setminus \{x\}$. Finally, we have $[c_1]_{\Pi_1^n(X \setminus \{x\},B)} = [c_2]_{\Pi_1^n(X \setminus \{x\},B)}$ and then i_* is one to one. □

Proof of Proposition 12.9 : Suppose that $T_n(x,X)$ and $T_{\overline{n}}(x,\overline{X})=1$. Then, following Proposition 12.10 and Proposition 12.11, $T_n(x,X)=1$ implies that Condition $i)$ of Definition 10.5 is satisfied. Furthermore, from Proposition 12.12 and Proposition 12.13, $T_{\overline{n}}(x,\overline{X})=1$ implies Condition $ii)$ of Definition 10.5. Finally, from Proposition 12.14 and Corollary 12.15, we have $T_n(x,X)=1$ and $T_{\overline{n}}(x,\overline{X})=1 \Rightarrow iii)$ and $iv)$. $\square$

Then, we achieve the proof of the main result of this part.

Proof of Theorem 16 : Following Definition 10.5, a simple voxel obviously satisfies the three conditions of Theorem 16. Now, from Proposition 12.1, a voxel which satisfies the three conditions of Theorem 16 is such that $T_n(x,X)=T_{\overline{n}}(x,\overline{X})=1$. Finally, from Proposition 12.9, if $T_n(x,X)=T_{\overline{n}}(x,\overline{X})=1$ then x satisfies the four conditions of Definition 10.5. $\square$

Conclusion of Part III

In this part, a new tool for studying topological properties of objects in $\mathbb{Z}^3$ has been introduced. This tool, the linking number, has the same properties as its continuous analogue. A proof of its most important properties is given with no need of the use of notions of the continuous case. Indeed, the proof given here need no more tools than those exposed in the first part of this thesis. The very few notions of digital topology which are used here show that some strong properties can be proved with the only use of the digital theoretical framework.

Furthermore, an application of the linking number to prove a new – because more concise – characterization of 3D simple points has been given (for $(n, \overline{n}) \in \{(6, 26), (26, 6)\}$). This new theorem shows the usefulness of the linking number in order to prove new theorems which involve the digital fundamental group in $\mathbb{Z}^3$.

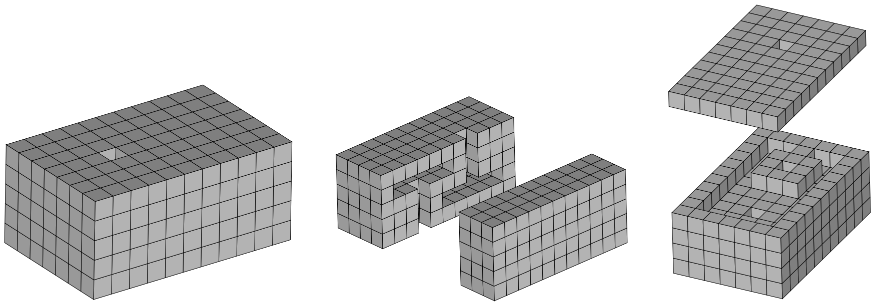
Now, even if the linking number is well defined for $(n, \overline{n}) \in \{(6+, 18), (18, 6+)\}$, it has not been used yet to provide a characterization of 3D simple points, similar to Theorem 16, for the latter couples of adjacency relations. This, because an open question remains about the existence of a simple closed curve, analogue to the curves $K_6(x, X, C)$ and $K_{26}(x, X, C)$ (Definitions 12.3 and 12.5), in this case. Nevertheless, further investigations should allow us to provide a simple process (such as "recursive deletion of $26-$bold voxels") which leads to the construction of the convenient curve, given a local configuration.

Conclusion and perspectives

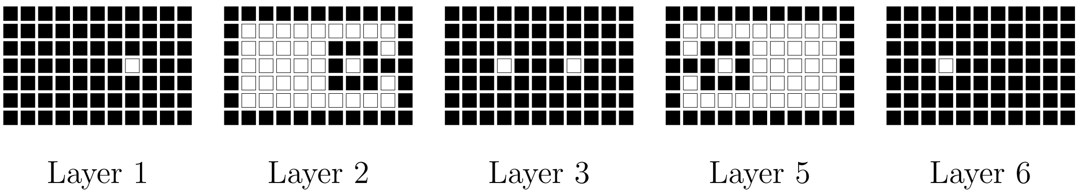
In this document, we have defined some new tools and proved several theorems which show that the digital fundamental group is a powerful tool for the characterization of topology preservation in a digital space. We may summarize this as follows :

- The digital fundamental group allows us to properly define simple surfels in digital surfaces, in other words, it fully characterizes topology preservation by removal of a unique spel in such a digital space.
- The digital fundamental group fully characterizes lower homotopy within digital surfaces (except in a very particular case).
- The digital fundamental group allows us to properly define simple voxels in $\mathbb{Z}^3$, in other words, it fully characterizes topology preservation by removal of a unique voxel. Furthermore, this characterization only involves the fundamental group of the object (not its complement).
- The digital fundamental group provides a theoretical criterion for lower homotopy for subsets of $\mathbb{Z}^3$.

However, it is shown that the digital fundamental group is not sufficient to characterize lower homotopy. Indeed, one could try to give some necessary and sufficient conditions for an objet $Y \subset X$ to be lower $n-$homotopic to X when X and Y are subsets of $\mathbb{Z}^3$. Today, it appears that such a condition will be very hard to find using the tools we have at our disposal. Indeed, the digital fundamental group which is very useful to formalize a global characterization of 3D simple points (see Chapter 10 and Chapter 12) shows some limitations for the characterization of lower homotopy. The fact is that the natural conditions of Definition 10.5 are not sufficient since the object Z depicted in Figure 12.11 (thanks to T.Y. Kong for this example which may also be found as an exercise page 189 in [43]) has the property that any element of $A_6^B(Z)$ is reducible in Z for any voxel $B \in Z$. Then, let x be any voxel of Z, it follows that the morphism $i_* : \Pi_1^6(\{x\}, x) \longrightarrow \Pi_1^6(Z, x)$ induced by the inclusion of $\{x\}$ in Z is an isomorphism. However, the set $\{x\}$ is obviously not lower $6-$homotopic to Z since this latter set has no $6-$simple voxel according to Theorem 16.



(a) A 3D outside view of the house-with-two-rooms.

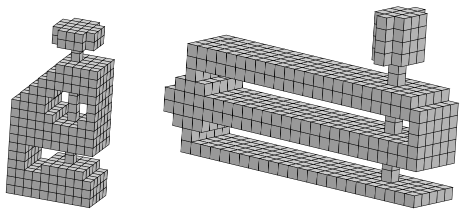


(b)

Figure 12.11: The “Bing’s house-with-two-rooms”.

Now, we could also try to find some necessary and sufficient conditions for two objects of $\mathbb{Z}^3$ to be equivalent up to a symmetric homotopic deformation. In this purpose, the digital fundamental group may appear as a useful tool. Intuitively, the object Z previously introduced and its background are both characterized by the fact that for any voxel $B \in Z$ and any $B' \in \overline{Z}$, the groups $\Pi_1^6(Z, B)$ and $\Pi_1^{26}(\overline{Z}, B')$ are trivial. Then, since it is obviously possible to sequentially add and remove simple voxels to this set and finally obtain a set reduced to a single voxel, we could hope that the digital fundamental group characterizes symmetric homotopy (this latter notion is once again illustrated in Figure 12.12). Obviously, the answer to this hope is no.

Indeed, let us suppose that an algorithm exists which always answer in a reasonable time to the following problem :



(a) (b)

Figure 12.12: Two subsets of $\mathbb{Z}^3$ which are the same up to a symmetric homotopic deformation.

`SH3D` : **Data :** $X \subset \mathbb{Z}^3$ and $Y \subset \mathbb{Z}^3$

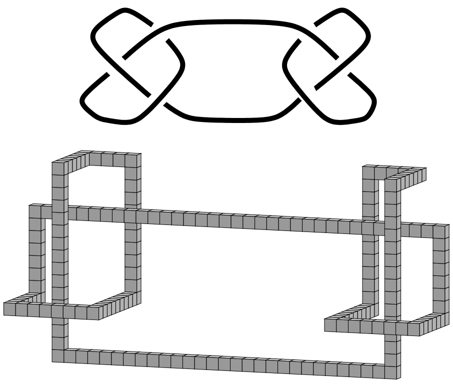
Result : YES *if* X *and* Y *are symmetrically homotopic*

NO *otherwise.*

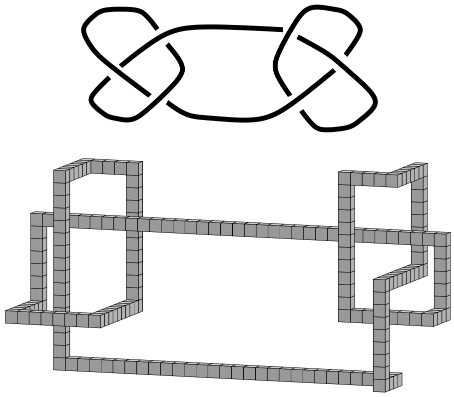
Today, no proof has been given which states that such a problem is decidable. Indeed, even a naive method which would check all the possible sequences of deletion or addition of simple voxels between the two considered objects is not proved to end for any input data.

Then, consider the problem of deciding whether two knots in $\mathbb{R}^3$ can or cannot be deformed one into each other by an ambient continuous deformation. This problem will have a solution when one will provide a complete invariant of the knots, which today has not been found. Now, it is readily seen that given two knots in $\mathbb{R}^3$, one can find a digitalization step under which a digital image of each knot can be given in such a way that the knot problem becomes a case of the symmetric homotopy problem (see Figure 12.13). Rigorously, we should prove that any ambient continuous deformation of a knot can be achieved by a sequence of insertion/deletion of simple voxels. Note that this latter property is very dependent to the digitalization step. Furthermore, we should also prove that symmetric homotopy cannot link together two objects whose polygonal analogues are not the same up to a continuous deformation. Nevertheless, the sketch given here may convince the reader that the symmetric homotopy problem is at least as difficult as the knot problem.

However, a classical result of knot theory, which shows that the knot group is not a complete invariant for knots, is that the digital (digital) fundamental groups of the complement of the objects X_1 and X_2 of Figure 12.13 are isomorphic whereas this two knots are not equivalent. In other words, the objects X_1 and X_2 are not symmetrically homotopic.



(a) X_1



(b) X_2

Figure 12.13: For any $B_1 \in \overline{X_1}$ and any $B_2 \in \overline{X_2}$, the fundamental groups $\Pi_1^n(\overline{X_1}, B_1)$ and $\Pi_1^n(\overline{X_2}, B_2)$ are isomorphic.

In [78], Nakamura and Rosenfeld stated the link between polygonal knots in $\mathbb{R}^3$ and digital knots in $\mathbb{Z}^3$. This latter paper can be seen as a first attempt to make a link between knot theory and the characterization of symmetric homotopy in $\mathbb{Z}^3$.

Notations

Notation	Name	Page
$\overline{X}$	complement of the set X	17
$Card(X)$	cardinal number of the set X	
$(x_0, \ldots, x_p)$ or $(x_i)_{i=0,\ldots,p}$	$p-$uplet	
$N_{\mathcal{R}}(x)$	$\mathcal{R}-$neighborhood of x	18
$G_n(x, X)$ for $n \in \{6, 6+, 18, 26\}$	geodesic $n-$neighborhood ($\mathbb{Z}^3$)	158
$G_e(x, X)$, $G_v(x, X)$	e_x-adjacency graph, v_x-adjacency graph	83
$\mathcal{R}_4$, $\mathcal{R}_8$	Adjacency relations in $\mathbb{Z}^2$	21
$\mathcal{R}_6$, $\mathcal{R}_{18}$, $\mathcal{R}_{26}$	Adjacency relations in $\mathbb{Z}^3$	22
$\pi_1.\pi_2$	paths catenation	20
π^*	set of the spels of a path	20
$\sim_{\mathcal{R}}$	relation of elementary $\mathcal{R}-$deformation	38
$\simeq_{\mathcal{R}}$	$\mathcal{R}-$homotopy relation	38
$A^B_{\mathcal{R}}(X)$	set of closed $\mathcal{R}-$paths from B to B in X	42
$\Pi^{\mathcal{R}}_1(X, B)$	quotient set of $A^B_{\mathcal{R}}(X)$ following $\simeq_{\mathcal{R}}$	42
$[c]_{\Pi^{\mathcal{R}}_1(X,B)}$	homotopy class of c in X following $\simeq_{\mathcal{R}}$	43
$\mathcal{I}_{\pi,c}$	Intersection number of the paths π and c	97
$Left_{\pi}(k)$, $Right_{\pi}(k)$	left an right local sets of a path	94
$L_{\pi,c}$	Digital linking number of the paths π and c	172
$P_c(i)$	Projective movement of the path c at subscript i	171
$c_n(s)$	$n-$path associated with a path of border edgels	138

Appendix

Appendix A

Proof of Lemma 12.6

In this appendix, we provide the C source files of a program which investigates all the local $3 \times 3 \times 3$ configurations of points such that $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$; and which check that at least one of the sets $K_n(x, X, C)$ associated with each such configuration is a simple closed $n-$curves for $n \in \{6, 26\}$ for some $\overline{n}-$connected component C of $G_n(x, \overline{X})$. This program is separated in several files. The file `config.h` contains the declaration of the data types and the few functions related to a local configuration (type `Config`). These latter functions are defined in the file `config.c`.

Description of the type `Config`

The type `Config` defined in `config.h` is an unsigned long integer where the bit of weight n is associated with a point of $N_{26}(x) \cap X$ following the convention of Figure A.1. As an illustration, we give in Figure A.2 the meaning of the bit mask "MASK_CUBE0" of the file `config.h`. Thus, operation such that "add to the configuration `cnf_a` the 6−neigbords of the point number i in the configuration `cnf_b`" can be achieve by very simple bitwise operation. Similarly, checking if all the points of a configuration belong to a 2×2×2 cube is done using the concise condition found in the file `config.c` (function In_2x2x2Cube()).

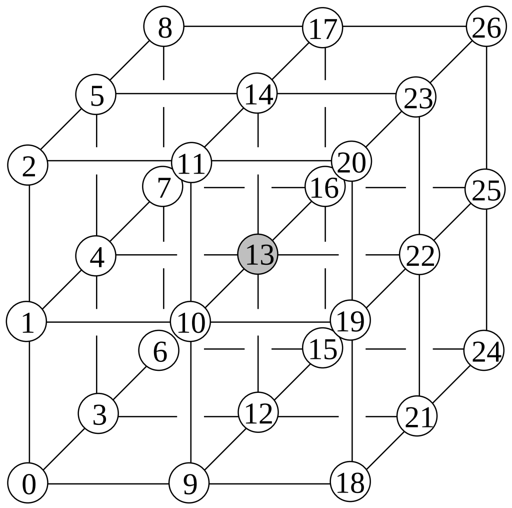


Figure A.1: Convention for the parameterization of $N_{26}(x)$.

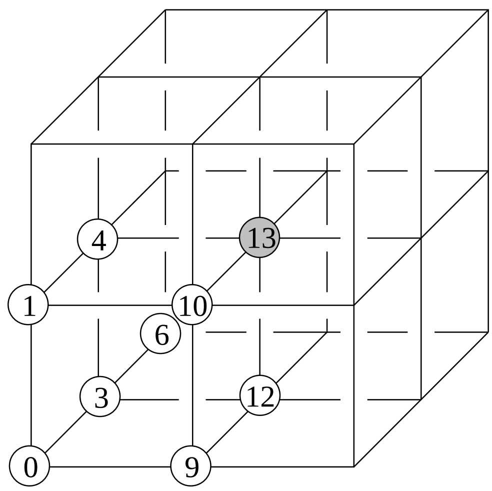


Figure A.2: This $2 \times 2 \times 2$ cube is represented as the integer $2^0 + 2^1 + 2^3 + 2^4 + 2^9 + 2^{10} + 2^{13}$ (= "MASK_CUBE0" in file `config.h`).

File : `Makefile`

```
CC = gcc -Wall

proof: proof.c config.o
        gcc -o proof proof.c config.o

config.o: config.h config.c
        gcc -c config.c
```

File : config.h

```
#ifndef CONFIG_H
#define CONFIG_H

/* *********************************************************************
 *  FILE : config.h
 *  AUTHOR : Sebastien FOUREY
 *  DESCRIPTION : definition of several constants and arrays for
 *  the manipulation of local 3x3x3 configurations.
 *
 *  Declarations of the following functions
 *
 * int cnfCardinal(Config);
 * Config cnfComplement(Config);
 * int cnfNbConnectedComponents(Config, int list[][32])
 * Config cnfG6(Config cnf);
 * Config cnfG26(Config cnf);
 * Config FindConnectedComponent(Config, const Config []);
 * int SimpleClosed26Curve(Config);
 * int SimpleClosed6Curve(Config config);
 *
 ***********************************************************************
*/

#define CENTER 13
#define END 127

#define MASK_CONFIG 0x07FFFFFF
#define MASK_CENTER 0x00002000

#define MASK_N26 0x07FFDFFF
#define MASK_N18 0x2EBDEBA
#define MASK_N6 0x415410

#define MASK_0 0x1
#define MASK_1 0x2
#define MASK_2 0x4
#define MASK_3 0x8
#define MASK_4 0x10
#define MASK_5 0x20
#define MASK_6 0x40
#define MASK_7 0x80
#define MASK_8 0x100
#define MASK_9 0x200
#define MASK_10 0x400
#define MASK_11 0x800
#define MASK_13 0x2000
#define MASK_14 0x4000
```

```
#define MASK_15 0x8000
#define MASK_16 0x10000
#define MASK_12 0x1000
#define MASK_17 0x20000
#define MASK_18 0x40000
#define MASK_19 0x80000
#define MASK_20 0x100000
#define MASK_21 0x200000
#define MASK_22 0x400000
#define MASK_23 0x800000
#define MASK_24 0x1000000
#define MASK_25 0x2000000
#define MASK_26 0x4000000

#define MASK_CUBE0 0x361B
#define MASK_CUBE1 0x6C36
#define MASK_CUBE2 0x1B0D8
#define MASK_CUBE3 0x361B0
#define MASK_CUBE4 0x6C3600
#define MASK_CUBE5 0xD86C00
#define MASK_CUBE6 0x361B000
#define MASK_CUBE7 0x6C36000

typedef unsigned long Config;

/* The6Neighbors[i] is a configuration which contains all the points */
/* of N26(x) which are 6-adjacent to the point number i              */
static const Config The6Neighbors[27] = {
      0x20a,     0x415,    0x822,     0x1051,     0xaa,    0x4114,
      0x8088,  0x10150,  0x200a0,    0x41401,   0x80a02,  0x104404,
    0x208208, 0x415410, 0x820820, 0x1011040, 0x2028080, 0x4014100,
    0x280200, 0x540400, 0x880800, 0x1441000, 0x2a80000, 0x4504000,
   0x2208000, 0x5410000, 0x2820000};

static const Config The26Neighbors[27] = {
     0x161a,    0x5e3d,   0x4c32,    0x196d3,   0x3dfef,    0x34d96,   0x19098,
    0x3d178,   0x340b0, 0x6c141b, 0xfc5a3f,   0xd84436, 0x36d86db, 0x7ffdfff,
  0x6db0db6, 0x36110d8, 0x7e2d1f8, 0x6c141b0, 0x681600, 0xf45e00, 0xc84c00,
  0x34d9600, 0x7bfde00, 0x65b4c00, 0x2619000, 0x5e3d000, 0x2c34000 };


static const int List6Neighbors[27][32] =
{{ 1, 3, 9, END },
 { 0, 2, 4, 10, END },
 { 1, 5, 11, END },
 { 0, 4, 6, 12, END },
 { 1, 3, 5, 7, END },
 { 2, 4, 8, 14, END },
 { 3, 7, 15, END },
```

```
 { 4, 6, 8, 16, END },
 { 5, 7, 17, END },
 { 0, 10, 12, 18, END },
 { 1, 9, 11, 19, END },
 { 2, 10, 14, 20, END },
 { 3, 9, 15, 21, END },
 { 4, 10, 12, 14, 16, 22, END },
 { 5, 11, 17, 23, END },
 { 6, 12, 16, 24, END },
 { 7, 15, 17, 25, END },
 { 8, 14, 16, 26, END },
 { 9, 19, 21, END },
 { 10, 18, 20, 22, END },
 { 11, 19, 23, END },
 { 12, 18, 22, 24, END },
 { 19, 21, 23, 25, END },
 { 14, 20, 22, 26, END },
 { 15, 21, 25, END },
 { 16, 22, 24, 26, END },
 { 17, 23, 25, END }};


static const int List26Neighbors[27][32] =
{{ 1, 3, 4, 9, 10, 12, END },
 { 0, 2, 3, 4, 5, 9, 10, 11, 12, 14, END },
 { 1, 4, 5, 10, 11, 14, END },
 { 0, 1, 4, 6, 7, 9, 10, 12, 15, 16, END },
 { 0, 1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, END },
 { 1, 2, 4, 7, 8, 10, 11, 14, 16, 17, END },
 { 3, 4, 7, 12, 15, 16, END },
 { 3, 4, 5, 6, 8, 12, 14, 15, 16, 17, END },
 { 4, 5, 7, 14, 16, 17, END },
 { 0, 1, 3, 4, 10, 12, 18, 19, 21, 22, END },
 { 0, 1, 2, 3, 4, 5, 9, 11, 12, 14, 18, 19, 20, 21, 22, 23, END },
 { 1, 2, 4, 5, 10, 14, 19, 20, 22, 23, END },
 { 0, 1, 3, 4, 6, 7, 9, 10, 15, 16, 18, 19, 21, 22, 24, 25, END },
 { 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15, 16, 17, 18,
   19, 20, 21, 22, 23, 24, 25, 26, END },
 { 1, 2, 4, 5, 7, 8, 10, 11, 16, 17, 19, 20, 22, 23, 25, 26, END },
 { 3, 4, 6, 7, 12, 16, 21, 22, 24, 25, END },
 { 3, 4, 5, 6, 7, 8, 12, 14, 15, 17, 21, 22, 23, 24, 25, 26, END },
 { 4, 5, 7, 8, 14, 16, 22, 23, 25, 26, END },
 { 9, 10, 12, 19, 21, 22, END },
 { 9, 10, 11, 12, 14, 18, 20, 21, 22, 23, END },
 { 10, 11, 14, 19, 22, 23, END },
 { 9, 10, 12, 15, 16, 18, 19, 22, 24, 25, END },
 { 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, END },
 { 10, 11, 14, 16, 17, 19, 20, 22, 25, 26, END },
 { 12, 15, 16, 21, 22, 25, END },
```

```
 { 12, 14, 15, 16, 17, 21, 22, 23, 24, 26, END },
 { 14, 16, 17, 22, 23, 25, END }};

int cnfCardinal(Config);
/* Returns the number of point in the configuration */

inline Config cnfComplement(Config);
/* Returns the complement of a configuration */

Config cnfG6(Config);
/* Returns G_{6}(x,X) */

Config cnfG26(Config);
/* Returns G_{26}(x,X) */


Config FindConnectedComponent(Config config, const Config TheNeighbors[]);
/* Returns an n-connected component of the configuration
   neighbrs[i] is a configuration which contains the points of
   N_{26}(x) which are n-adjacent to the point i
   Returns 0 is the configuration is empty
*/

int SimpleClosed26Curve(Config);
/* Checks if the configuration is a simple closed 26-curve */

int SimpleClosed6Curve(Config);
/* Checks if the configuration is a simple closed 6-curve */

int cnfNbConnectedComponents(const Config config, const int list_neighbors[27][32]);
/* Counts the number of n-connected components in the configuration */

#endif
```

File : config.c

```
#include "config.h"
/* ***********************************************************************
 *  FILE : config.c
 *  AUTHOR : Sebastien FOUREY
 *  DESCRIPTION : definition of the functions declared in config.h
 ***********************************************************************
*/

int cnfCardinal(Config config)
{
  int result=0;
  while (config)
  {
      if (config & 1) result++;
      config>>=1;
  }
  return result;
}

inline Config cnfComplement(Config config)
{  return     (~config)&MASK_N26; }

Config cnfG6(Config config)
{
  Config result=0;
  Config add=0;

  result = config & MASK_N6;

  if (result & MASK_4) add |= The6Neighbors[4];
  if (result & MASK_10) add |= The6Neighbors[10];
  if (result & MASK_12) add |= The6Neighbors[12];
  if (result & MASK_14) add |= The6Neighbors[14];
  if (result & MASK_16) add |= The6Neighbors[16];
  if (result & MASK_22) add |= The6Neighbors[22];
  result |= (config & add);
  return result;
}

Config cnfG26(Config cnf)
{  return cnf & MASK_N26; }

Config FindConnectedComponent(Config config, const Config TheNeighbors[])
{
  Config old,grow,add;
  int i;
```

```
  if (!config) return 0;
  i=0;                                    /* Find a starting point   */
  while ( !(config & (1<<i)) ) i++;

  grow = 1<<i;                            /* Grow the component associated with */
  do
  {                                       /* point number i until stagnation   */
    old = grow;
    add=0;                                /* Add to grow the 6-beighbords in   */
    for (i=0; i<27 ; i++)                 /* configuration of the points of grow */
      if (grow & (1<<i))
        add |=(config & (TheNeighbors[i]));
    grow |= add;
  } while (grow != old);

  return grow;
}

int SimpleClosed26Curve(Config config)
{
  int i;
  for (i=0; i<27 ; i++)
  {
      if ( ( config & (1<<i)) &&
           ( cnfCardinal( config & The26Neighbors[i]) != 2))
        return 0;
  }
  return 1;
}

int SimpleClosed6Curve(Config config)
{
  int i;
  for (i=0; i<27 ; i++)
  {
    if ( ( config & (1<<i)) &&
         ( cnfCardinal(config & The6Neighbors[i]) != 2))
        return 0;
  }
  return 1;
}

int cnfNbConnectedComponents(const Config config, const int list_neighbors[27][32])
{
  Config not_seen,mask;
  int n=0;
  int result=0;
  unsigned int queue[32];
  unsigned int * q_head = queue;
```

```
  unsigned int * q_end = queue;
  const int * neighbor;

  not_seen = MASK_N26 & config;

  while (not_seen)
  {
      n=0; result++;
      q_head=q_end=queue;
      mask=1;
      while (!(not_seen & mask)) { n++; mask<<=1; }
      *q_end = n;
      q_end++;
      not_seen^=mask;
      while (q_head<q_end)
      {
          neighbor = list_neighbors[*q_head];              /* dequeue a point */
          q_head++;

          while (*neighbor != END)
          {
              mask = 1<<(*neighbor);
              if (not_seen & mask)     /* enqueue neighbor if not already visited */
              {
                  *q_end=*neighbor;
                  q_end++;
                  not_seen ^= mask;     /* Mark neighbor as visited             */
              }
              neighbor++;
          }
      }
  }
  return result;
}
```

File : proof.c

```
#include "config.h"
/* ********************************************************************
 *                                                                    *
 *   FILE : proof.c                                                   *
 *   AUTHOR : Sebastien FOUREY                                        *
 *   DESCRIPTION : Automatic proof                                    *
 *                                                                    *
 *********************************************************************
*/

/* Find all the 6-connected components C of G_6(x,\overline{X}) and check *
 * for each one if the associated set K is a simple closed 26-curve.      *
 * Return 0 if no such K is a simple closed 26-curve                       *
 * Return 1 otherwise                                                      */
int FindK26Candidates(Config configuration);

/* Find all the 26-connected components C of G_26(x,\overline{X}) and check *
 * for each one if the associated set K is a simple closed 6-curve.         *
 * Return 0 if no such K is a simple closed 6-curve                          *
 * Return 1 otherwise                                                        */
int FindK6Candidates(Config configuration);

/*  Check if all the points of the configuration belong to *
 *  a 2x2x2 cube                                            */
int In_2x2x2Cube(Config config);

/*  MAIN FUNCTION */
int main()
{
  Config config;
  unsigned long n;

  /* Proof of the case (26,6) */
  printf("Exploring config. : T26(x,X)=1 et T6(x,overline{X})>=2\n");
  for (config= 0 ; config <= 0x07FFFFFF ; config++)
  {
      if (!(config & MASK_CENTER) &&
          (cnfNbConnectedComponents(cnfG26(config),
                                    List26Neighbors) == 1) &&
          (cnfNbConnectedComponents(cnfG6(cnfComplement(config)),
                                    List6Neighbors) >= 2))
      {
           n++;
           if (!FindK26Candidates(config))
           {
              printf("Error: No s.c. 26-curve found %lu (%lx)\n",config,config);
              exit(-1);
```

```
        }
      }
    }
    printf( " %lu configurations w. T26(x,X)=1 and T6(x,overline{X})>=2 \n",n);

    /* Proof of the case (6,26) */
    printf(" Exploring config. : T6(x,X)=1 and T26(x,overline{X})>=2\n");
    n=0;
    for (config=0 ; config <= 0x07FFFFFF ; config++)
    {
      if (!(config & MASK_CENTER) &&
          (cnfNbConnectedComponents( cnfG6(config),
                                     List6Neighbors)==1) &&
          (cnfNbConnectedComponents(cnfG26(cnfComplement(config)),
                                    List26Neighbors)>=2))
      {
          n++;
          if (!FindK6Candidates(config))
          {
              printf("Error: No s.c. 6-curve found %lu (%lx)\n",config,config);
              exit(-1);
          }
      }
    }
    printf( " %lu configurations w. T6(x,X)=1 and T26(x,overline{X})>=2\n",n);
}


/* Find all the 6-connected components C of G_6(x,\overline{X}) and check
   for each one if the associated set K is a simple closed 26-curve.
   Return 0 if no such K is a simple closed 26-curve
   Return 1 otherwise                                               */
int FindK26Candidates(Config configuration)
{
    Config K,K0,C;
    Config seen=0;
    Config G26 = configuration & MASK_N26;          /* G26(x,X)        */
    Config G6Comp = cnfG6(cnfComplement(configuration)); /* G6(x,\overline{X}) */
    int some_removal;
    int i,s;

    while (( C = FindConnectedComponent(G6Comp & ~seen, The6Neighbors)))
    /* C is a 6-connected component in G6(x,\overline{X}) which */
    /* has not been already tested  */
    {
      K0=0;  /*              Build the set K0 associated to cc_complement */
      for (i=0 ; i<27 ; i++)
      {
          if ((G26 & (1<<i))        /* The point "i" belongs to G26(x,X) */
```

```
           && (C & The6Neighbors[i])) /* and is 6-adjacent to C         */
         K0 = K0 | (1<<i);           /* so belongs to K0                */
    }

    /*                               Build the set K associated to C    */
    K = K0 & MASK_N18;               /* A voxel of N26(x)\setminusN18(x) */
                                     /* cannot be isolated in X and then */
                                     /* is obviously bold.               */

    do {                             /* Remove the bold voxels in K      */
      some_removal=0;
      for (i=0 ; i<27 ; i++)
      {
           if ( (K & (1<<i)) &&
                In_2x2x2Cube( (K & The26Neighbors[i]) & MASK_N26))
           {
               K = K ^ (1<<i);
               some_removal=1;
           }
      }
    }
    while (some_removal);

    if (SimpleClosed26Curve(K)) return 1;
    seen = seen | C;
  }
  return 0;                          /* No simple closed 26-curve found !!! */
}


/* Find all the 26-connected components C of G_26(x,\overline{X}) and check *
 * for each one if the associated set K is a simple closed 6-curve.       *
 * Return 0 if no such K is a simple closed 6-curve                       *
 * Return 1 otherwise                                                     */
int FindK6Candidates(Config configuration)
{
  Config K,K0;  /* The sets K and K0 */
  Config seen=0; /* Bitwise OR of all the connected C components of */
                 /* G26(x,\overline{X}) already investigated      */

  Config G6 = cnfG6(configuration);
  Config G26Comp = cnfG26((~configuration) & MASK_N26);
  Config C;
  int i,some_removal;

  /* Trouve une composante 26 connexe */
  while ((C=FindConnectedComponent( G26Comp & ~seen, The26Neighbors)))
  {
    K0=0;                          /* K0 is the set of voxel in G6(x,X) which */
```

```
      for (i=0 ; i<27 ; i++)    /* are 26-adjacent to the component C  */
      {
          if ( (G6 & (1<<i)) &&
               ( C & The26Neighbors[i]))
            K0 = K0 | (1<<i);
      }

      K = K0;
      if (SimpleClosed6Curve(K)) return 1; /* First test (optimization) */

      do {                              /* Remove the extremities in K */
          some_removal=0;
          for (i=0; i<27 ; i++)
          {
            if ((K & (1<<i)) && (cnfCardinal(K & The6Neighbors[i]) == 1))
            {
                  K = K ^ (1<<i);
                  some_removal = 1;
            }
          }
      }
      while (some_removal);

      if (SimpleClosed6Curve(K)) return 1;
      seen |= C;
  }
  return 0;                          /* No simple closed 6-curve found !!! */
}


/*   Check if all the points of the configuration belong to *
 *   a 2x2x2 cube                                            */
int In_2x2x2Cube(Config config)
{
  return ( config
           &&
           (  !(config & ~MASK_CUBE0)
           || !(config & ~MASK_CUBE1)
           || !(config & ~MASK_CUBE2)
           || !(config & ~MASK_CUBE3)
           || !(config & ~MASK_CUBE4)
           || !(config & ~MASK_CUBE5)
           || !(config & ~MASK_CUBE6)
           || !(config & ~MASK_CUBE7)));
}
```

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